NUMERICAL ANALYSIS

MATH (422)

LECTURE NOTES

BY

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PART 2

The Solution of Non-Linear Equations:

Algebraic equations can be divided into two class:

(i) Linear.

(ii) Non-Linear.

Each of these then sub-divides into two classes:

i.e. (i) One variable only.

(ii) More than one variable.

In a linear equation, all the variables present, occur only to the 1st power and no product of variables occur i.e. $(not\ xy, \frac{y}{x})$

Thus $3x + sy - z = 14$ is a linear equation with 3 variables.

$3x + 5yz - 7z = 14$ is not a linear equation.

When we have n independent variables $x_1, x_2, \ldots\ldots, x_n$ we need at least n equations (linear or non-linear) to find a unique solution (if one exists).

We now concentrate on non-linear equation in a single variable:

e. g:

$$3x^2 - 2x + 7$$

$$x^3 - 5x^2 - 4x - 17 = 0$$

$$(1+x)^{\frac{5}{2}} e^{-x^2} = \frac{1}{2}$$

In general there is no analytical method for solving non-linear equations and so we must use numerical methods in the sections that follow.

We shall develop and study five such methods. These five methods fall into two classes:

(a) Two-point methods and

(b) One-point methods.

(a) Two-points methods:

(1) The Method of Bisection:

Suppose that we wish to solve

$$f(x) = 0 \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

and that we have found two approximate values for the solution x_1, x_2 such that $f(x_1)f(x_2) < 0$ it follows that assuming $f(x)$ to be continuous over $< x_1, x_2 >$

there is a solution to (1) some where in the interval $<x_1, x_2>$

we therefore have the problem of how to choose a value x_3

such that

$$x_1 < x_3 < x_2 \ and \ f(x_3) = 0$$

The simplest method, from a computational point of view is

to take

$$x_3 = \frac{1}{2}(x_1 + x_2)$$

we now evaluate $f(x_3)$ if $f(x_1)f(x_3) < 0$

we choose a new point $x_4 = \frac{1}{2}(x_1 + x_3)$ whereas if

$f(x_1)f(x_3)>0$ then $f(x_2)f(x_3)<0$ and we choose

$$x_4=\frac{1}{2}(x_2+x_3)$$

and so on . At every stage we have had two points x_i, x_j such that $f(x_i)f(x_j)<0$ and we choose the next point to be

$$x_e=\frac{1}{2}(x_i+x_j)$$

and use this point and whichever of x_i, x_j causes $f(x)$ to have opposite sign to $f(x_e)$ as the two points for the following stage.

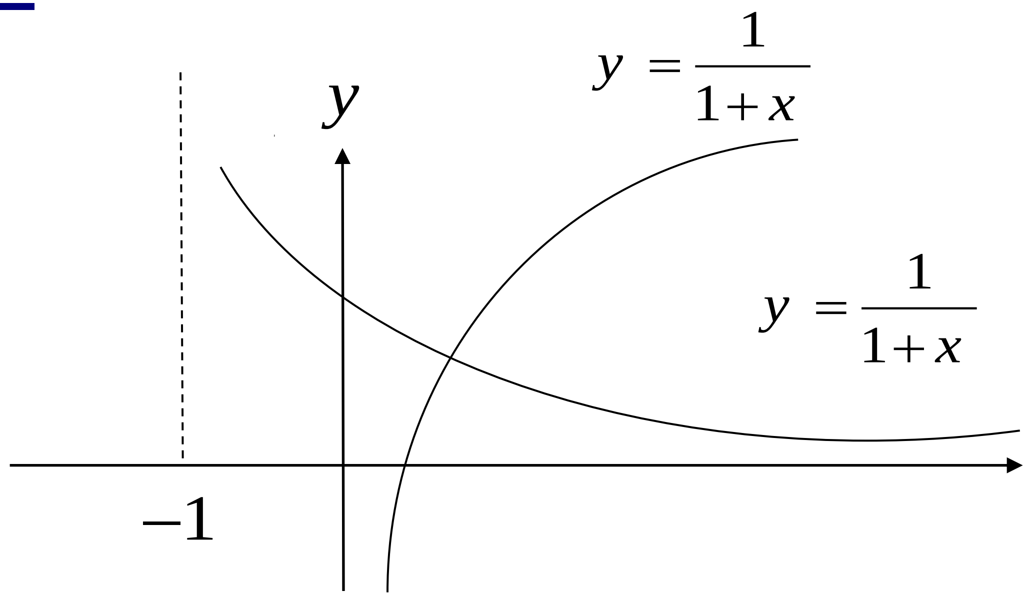
The process terminates when we reach a point x_n such that $|fx_n|$ is sufficiently small.

This method is known as the method of Bisection.

Example:

Slove $\dfrac{1}{1+x} = \log_e x$ by the method of Bisection?

Solution:



Put $f(x)=\frac{1}{1+x}-\log_c x$.

We want to solve $f(x)=0$

Try $x_1=1, \quad x_2=2$ $\quad x=1 \quad :$ $\qquad f(1)=\frac{1}{2}-\log_c l=\frac{1}{2}>0$

Try $x_1=1, \quad x_2=2$ $\quad x=2 \quad :$ $\qquad f(2)=\frac{1}{3}-\log_c 2=0.33-0.69<0$

so in the method of Bisection we start with $x_1=1$ *and* $x_2=2$

and we know $\exists$ a root between 1 and 2.

n	x_n	$f(x_n)$	
1	1	0.5 > 0	
2	2	-0.36 < 0	$x_3 = \frac{1}{2}(1+2) = 1.5$
3	1.5	-0.0055 < 0	$x_4 = \frac{1}{2}(1+1.5) = 1.2$
4	1.25	+0.3528 > 0	$x_5 = \frac{1}{2}(1.5+1.25) = 1$
5	1.375	+0.1026 > 0	$x_6 = \frac{1}{2}(1.5+1.375) =$
6	1.4375	+0.0474 > 0	$x_7 = \frac{1}{2}(1.5+1.4375) =$
7	1.46875	+0.0208 > 0	$x_8 = \frac{1}{2}(1.5+1.46875) =$
8	1.484375	+0.0075 > 0	
9	1.4921875	+0.002 > 0	Stabilizes as 1.49
10	1.496093575		2.d.p

This required 8 iterations to reach 2.d.p accuracy starting from an interval length 1.

Conclusion:

The method of Bisection works but convergence to the solution is very slow. The method is easy to program.

Example:

Find a solution of the equation

$$Sin\ x - \frac{1}{2}x = 0$$

in the interval $< \frac{1}{2}\pi, \pi >$.

Solution:

Put $f(x) = \sin x - \frac{1}{2}x$

We want to solve $f(x) = 0$

We have $f\left(\frac{1}{2}\pi\right) = 1 - \frac{1}{4\pi} > 0$

$$f(\pi) = \frac{-1}{2}\pi < 0$$

So, there is at least one solution of the equation in the interval.

So in the method of Bisetlion we start with $x_1 = \frac{\pi}{2}, x_2 = \pi$

(Assuming $\pi = 3$ for simplification.

n	x_n	$f(x_n)$	
1	$\frac{\pi}{2} = (1.5)$	$1 - \frac{\pi}{4} > 0$	
2	$\pi = (3.0)$	$-\frac{\pi}{2} < 0$	$x_4 = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$
3	$\frac{1}{2}\left(\frac{\pi}{2} + 2\frac{\pi}{2}\right) = 2.25$	$-0.34693 < 0$	$x_4 = \frac{1}{2}(1.5 + 2.25)$
4	1.875	$+0.01659 > 0$	$x_5 = \frac{1}{2}(2.25 + 1.875)$
5	2.0625	$-0.14972 < 0$	$x_6 = \frac{1}{2}(1.875 + 2.0625)$
6	1.96875	$-0.06253 < 0$	$x_7 = \frac{1}{2}(1.875 + 1.96875)$
7	1.921875	$-0.02194 < 0$	$x_8 = \frac{1}{2}(1.875 + 1.921875)$
8	1.898438	$-0.002415 < 0$	$x_9 = \frac{1}{2}(1.875 + 1.898-)$
9	1.886719	$0.007151 > 0$	$x_{10} = \frac{1}{2}(1.89 + 1.88-)$
10	1.8925785	$0.00238 > 0$	

n	x_n	$f(x_n)$
20	1.89549	+0.000002

The result is now correct to 5 d.p but convergence has been slow 18 iterations to reach the required accuracy.

The method of False Position:

The method of Bisection is very simple to use, easy to program for a computer and is certain to converge to a solution but it is unnecessarily slow in its convergence.

Can we speed this convergence?

Looking back, the example we had taken $x_3 = 1.5$

and found $x_3 = 1.5 = -0.0055$ and at that point we chose $x_4 = \frac{1}{2}(1+1.5)$ clearly this was a foolish thing to do since $f(1) = +0.5.$

Since 0.5 is approximately 100×0.0055 it would seem to be more sensible to choose x_4 to be that point which is $\frac{99}{100}$ this of the way from 1 to 1.5.

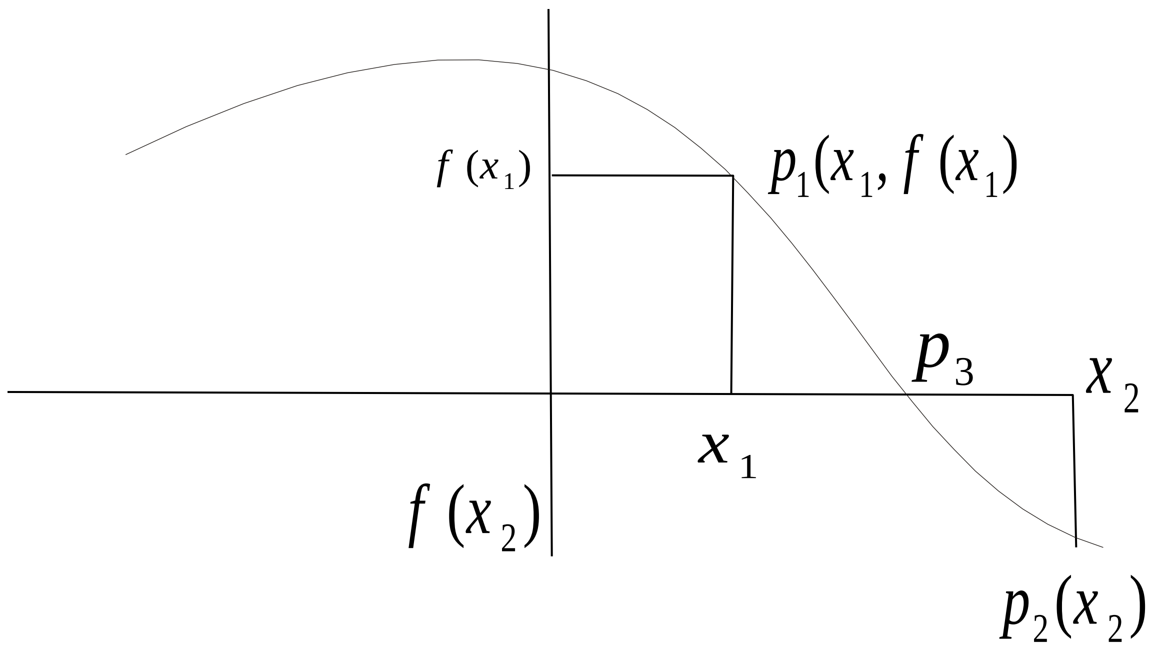
i.e. 1.495 this is nearly correct to 3 d.p already in fact.

Can we formalize this process?

Suppose we are trying to solve $f(x) = 0$ and have two approximations to the solution x_1 and x_2 such that $f(x_1) > 0$

and $f(x_2) < 0$

How should we choose our next approximation x_3



Let p_1 be the point $\left(x_1, f(x_1)\right)$ and p_2 be the point $\left(x_1, f(x_1)\right)$. Join p_1 to p_2 by a straight line. Since p_1 is above the $\chi -$ axis and p_2 below.

The line $p_1 p_2$ must cross the $\chi -$ axis at some point χ_3 between χ_1 and χ_2, χ_3 is then our new approximation to the root.

The equation of the chord $p_1 p_2$ is

$$\frac{y - f(x_1)}{x - x_1} = \frac{y - f(x_2)}{x - x_2}$$

and this line meets the line $y = 0$

Where

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

So, given two approximate values x_1, x_2 to the solution

We can construct a new approximate value x_3.

The question now arises. How should we construct the next approximation x_4? There are clearly two possibilities.

(1) to use x_1 and x_3

(ii) to use x_2 and x_3

How do we decide?

One reasonable way is as follows:

Evaluate $f(x_3)$ and choose x_1 and x_3 if $f(x_1)f(x_3) < 0$. Otherwise choose x_2 and x_3 (since $f(x_2)f(x_3)$ will be negative). This is the method of false position.

Example:

Solve $\frac{1}{1+x} = \log_e x$ by the method of false position?

Solution:

Try $x = 1$ $\quad f(1) = 0.5 > 0$

Try $x = 2$ $\quad f(2) = -3.36 < 0$

n	x_n	$f(x_n)$
1	1	0.5>0
2	2	-0.36<0
3	1.581	-0.071<0
4	1.5088	-0.0127<0
5	1.4962	-0.0023<0
6	1.4939	-0.00041<0
7	1.4935	-0.00008<0
8	1.4934	0.000003>0
9	1.493406	-0.00000/6
Correct to 5.d.p in 7 iterations.		

3. The Secant Method:

The alternative method is to ignore the sign of $f\,(x_3)$ and simply use x_3 and x_2 in the formula.

i.e. at each iteration use the two most recent values of x. Regardless of sign. This technique is simpler to program but it can't guarantee to converge to the root whereas the false rule will always coverage.

It can be proved however the secant method if it coverage's, will on average take only about 62% of the number of iteration of the false rule.

Example:

Solve $\dfrac{1}{1+x} = \log_e x$ by the secant method?

Solution:

n	x_n	$f(x_n)$
1	1	0.5
2	2	-0.36
3	1.581	-0.071
4	1.4781	+0.0128
5	1.4938	-0.00033
6	1.493405	-0.000008
Correct to 5.d.p is 4 iterations.		

Convergence of two point-Methods:

The general two point method for solving an equation $f(x)=0$ takes the form:

$$x_{n+1} = G(x_n, x_{n-k})$$

Thus for example, for Bisection.

$$G = \frac{x_n + x_{n-1}}{2}$$

Second rule:

$$G = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

For false Rule:

$$G = \frac{x_n f(x_{n-k}) - x_{n-k} f(x_n)}{f(x_{n-k}) - f(x_n)}$$

where,

$$f(x_n) f(x_{n-k}) < 0$$

An important question is: does any particular method converge to the true solution? It so under what conditions and how fast?

Convergence of the method of Bisection:

At each stage we have two approximations to the solution x_n, x_{n-1} which have the property that $f(x_n)f(n_{n-1})<0$. So there is a root always in the interval

$$<x_n,\ x_{n-1}>$$

The next point x_{n+1} is $\dfrac{x_n+x_{n-1}}{2}$ and our new root lies either in the interval

$$<x_n, x_{n+1}> or\ \ in < x_{n+1}, x_{n-1}>$$

and the sign of $f(x_{n+1})$ tells us which interval it is.

Therefore, if when we started, we know that the root lay in

the interval $< x_o, x_1 >$ of length $|x_1 - x_1| = d \ (say)$

after the 1st iteration we would know which of the intervals

$< x_2, x_1 >$ and $< x_2, x_0 >$ contained the root; these intervals

are of length $\frac{1}{2} d$; clearly after n iterations we will know

that the root lies in a particular interval of length $\frac{d}{2^n}$; ,

since d is fixed $\frac{d}{2^n} \to 0 \ as \ n \to \infty$.

So, the process converges and furthermore we can work out as many iterations we need to get the root correct to say, m.d.p for after n iterations the root with an error of $\frac{d}{2^n}$ and if this is to be correct to m d.p we must choose n so that $\frac{d}{2^n} < \frac{1}{2} \times 10^{-m}$

$$i.e/ \qquad 2^{n-1} > 10^m d$$

$$i.e/ \qquad n > 1 + m \ \log_2 10 + \log_2 d$$

$$\square \qquad 1 + 3\frac{1}{3} m + \log_2 d$$

$$Since \quad \log_2 10 \,\square\, 3\frac{1}{3}.$$

for example when we had $f(x)=\dfrac{1}{1+x}-\log_e x$ we had

$<x_o,x_1>=<1,2>$ so $d=1$, So for 2 d.p accuracy we would expect to need n iterations.

Where

$$n \approx 1+\left(3\frac{1}{3}\right)2=7\frac{2}{3}$$

In fact we needed 8, if we now wish to go to 5 d.p accuracy the total numbers of iteration we would need be about.

$$1+\left(3\frac{1}{3}\right)5=17\frac{2}{3}$$ i.e/about 18.

Thus we have proved that the Bisection Method converges and if ε_{n+1} is the error in the approximation after $n+1$ steps then $\varepsilon_{n+1} \square \frac{1}{2}\varepsilon_n$ which then yields the formula* above.

For the method of false rule it can be shown that convergence will always occur and the errors in the solution at two consecutive iterations, ε_n and ε_{n+} are related by $\varepsilon_{n+1}, \square\ k.\varepsilon_n . Where\ |K| <|$

where the value of K depends upon particular function *f(x).* If */ k /* is small convergence is fast, if | k | is nearly = 1 convergence is slow.

These two methods both have the propects that the error at consecutive stages are related by a formula of the type $\varepsilon_{n+1} \approx A\varepsilon_n$ where *A* is some constant.

In each cases we say that the process converges linearly.

There are also processes where the errors at consecutive stages $\varepsilon_n, \varepsilon_{n+1}$ are related by a formula of the type $\varepsilon_{n+1} \approx A\varepsilon_n^p$ in which we say that the method converges with power P.

It can be proved that the 2nd method converges with power $\frac{\sqrt{5}+1}{2}$ the bigger the value of P, the faster the method will converge.

(b) The one point methods:

We shall discuss two such methods:

1) Newton-Raphson Method with two variations.

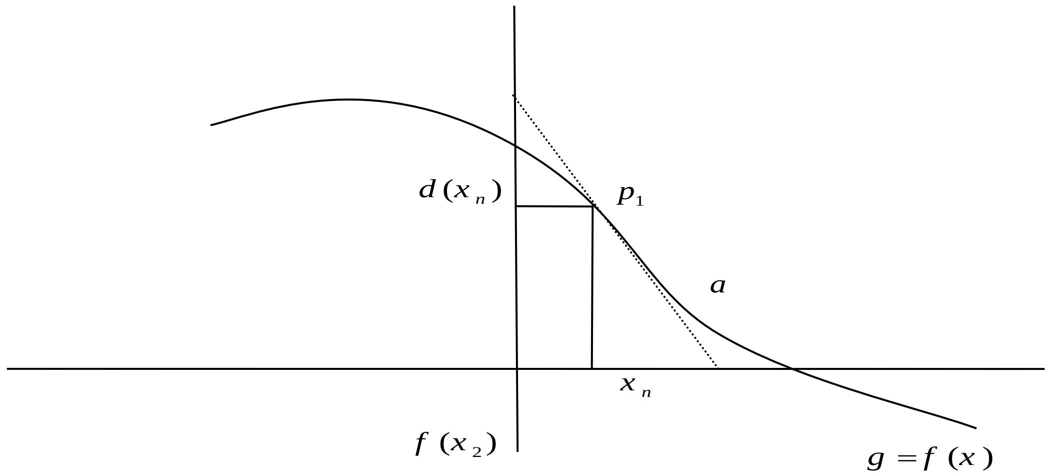
(i) For multiple roots.

(ii) Stephenson's Methods.

2) General Iterative Method.

1. Newton-Raphson Method:

If, in the secant Method, we let the two starting values x_1, x_2 become arbitrarily close we eventually replace the secant joining the points $P.(x_1, f(x)).p_2((x_2, f(x_2))$ by the tangent at P_1.



Let T be the tangent at $p_1(x_n . f(x_n)$ and let T cross the line y = 0 at a then a is $(x_{n+1,}\ 0)$ and x_{n+1} is our new approximation. We can find the value of x_{n+1} easily for:

$$\frac{f(x_n) - 0}{x_n - x_{n+1}} = f'(x_n).$$

from which we deduce that $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

this is the Newton-Raphson formula.

Example :

$$Solve\,\frac{1}{1+x}=\log_e x$$

By the Newton – Raphson method?

Solution :

Start at $x_1 = 1$

$$f(x)=\frac{1}{1+x}-\log_e x$$

$$f'(x)=\frac{-1}{(1+x)^2}-\frac{1}{x}$$

Newton – Raphson Method gives.

$$x_{n+1} = x_n + \frac{\left(\frac{1}{1+x_n} - \log x_n\right)}{\left(\frac{1}{(1+x_n)^2} + \frac{1}{x_n}\right)}$$

n	x_n =
1	1
2	1.4 = $(1+\left(\frac{1}{2}-0\right)/\left(\frac{1}{4}+1\right))$
3	1.4903 = $(1.4+\frac{1}{2.4}-\log 1.4)/(\frac{1}{5.76}+\frac{1}{1.4})$
4	1.49340 = Correct to 5.d.p

Thus the Newton Raphson gets 5.d.ps in 3 iterations compared with 16 by the method if Bisection.

Example:

Find $2^{\frac{1}{3}}$ by the Newton Raphson Method?

Solution:

$2^{\frac{1}{3}}$ is the root of $x^3 - 2 = 0$

so, in the N-R we put

$$f(x) = x^3 - 2$$

$$f'(x) = 3x^2$$

$$x_{n+1} = x_n - \frac{\left(x_n^3 - 2\right)}{3x_n^2}$$

$$= \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$$

We start at $x_1 = 1$

$$x_2 = \frac{4}{3} = 1.3333$$

$$x_3 = 1.26389 \quad (correct\ \ to\, 2d\,.p\,)$$

$$x_4 = 1.2599335\ (correct\ \ to\ 4d\,.p\,)$$

$$x_5 = 1.25992105\ (correct\ \ to\ \ 8d\,.p\,)$$

i.e

$$\varepsilon_3 = 3.9 \times 10^{-3}$$

$$\varepsilon_4 = 1.245 \times 10^{-5}$$

$$\varepsilon_5 = 1.06 \times 10^{-10}$$

Theorem:

If ρ is the exact solution of $f(x) = 0$ and $x_1 = \rho + \varepsilon_1$ the Newton Rophson method will converage provided that ε_1 is sufficiently small and

$$\left| \frac{1}{2} \varepsilon_1^2 \frac{f''(\rho)}{f'(\rho)} \right| < 1$$

Proof:

Let $x_1 = \rho + \varepsilon_1$ where ρ is the exact solution of $f(x) = 0$

Then our next estimate of ρ

$$x_2 = x_1 - \frac{f(x_1)}{f`(x_1)}$$

$$= \rho + \varepsilon_1 - \frac{f(\rho + \varepsilon_1)}{f`(\rho + x_1)}$$

and so, since $f(\rho) = 0 \; if \quad \varepsilon_1$ is sufficiently small,

$$x_2 = \rho + \varepsilon_1 \frac{\varepsilon_1 f'(\rho) + \frac{1}{2}\varepsilon_1^2 f^{1}(\rho)}{f'(\rho) + \varepsilon_1 f''(\rho)}$$

which simplifies, on neglecting ε_1^3 and higher term.

$$x_2 = \rho + \frac{1}{2}\varepsilon_1^2 \frac{f''(\rho)}{f'(\rho)}$$

Thus

$$\left| x_2 = \rho \right| = \left| \frac{1}{2}\varepsilon_1^2 \frac{f''(\rho)}{f'(\rho)} \right|$$

and so $|x_2 - \rho| < |x_1 - \rho| = |\varepsilon_1|$

provided $\left| \frac{1}{2} \varepsilon_1^2 \frac{f''(\rho)}{f'(\rho)} \right| < 1$

this proves the theorem.

We will have

$$x_n = \rho + \varepsilon_n$$

$$i.e / x_n - \rho = \varepsilon_n .$$

so that the error will decrease quadratically. Thus if our first estimate is accurate to 1 d.p. our 2nd should be accurate to 2 d.p our 3rd to 4 d.p. Our 4th to 8 d.p and so on.

Example:

Use the Newton Raphson Method to find $\sqrt{2}$ to 4.d.p starting from $x=1$?

Solution:

In this case $f(x)=x^2-2$

$$f(x)=x^2-2$$

$$f`(x)=2x$$

$$N-R\ formula:$$

$$x_{n+1}=x_n-\frac{\left(x_n^2-2\right)}{2x_n}$$

$$=\frac{1}{2}\left(x_n+\frac{2}{x_n}\right)$$

Hence we have the successive approximations.

n	x_n	ε_n
1	1	0.4142
2	1.5	0.0858 (1 d.p)
3	1.4167	0.0025 (2 d.p)
4	1.4142	0.000014 (4.d.p)

(1) The Newton Raphson method may fail, or at best converage slowly if the function has amultiple root or two roots very close together.

To over come this difficulty modified versions of the Newton Raphson method have been developed, one of which we now examine.

Suppose that $f(x)$ has a zero at $x=\alpha$ of multiplicity *K*. then

$$f(\alpha)=f'(\alpha)=\ldots \qquad \ldots=f^{k-1}(\alpha)=0$$

$$but \;\; f^{k}(\alpha)\neq 0$$

We modify the Newton Raphson formula

$$x_{n+1}=x_{n-}\frac{f(x_n)}{f'(x_n)}$$

by introducing a parameter λ as a factor of the second term vis.

$$x_{n+1} = x_n - \frac{\lambda f(x_n)}{f'(x_n)}$$

and we carryout an analysis to find the best value for λ.

Let

$$x_n = \alpha + \varepsilon_n ;\ then$$

$$f(x_n) = f(\alpha + \varepsilon_n) \approx f(\alpha) + \varepsilon_n f'(\alpha) + \dots$$

$$+ \frac{\varepsilon_n^k}{k!} f^k(\alpha) + \frac{\varepsilon_n^{k+1}}{(k+1)!} f^{k+1}(\alpha)$$

and $f'(x_n) = f'(\alpha + \varepsilon_n) \approx f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots$

$$+ \frac{\varepsilon_n^{k-1} f^{k}(\alpha)}{(k+1)!} + \frac{\varepsilon_n^{k}}{k!} f^{k+1}(\alpha)$$

since α is a zero of multiplicity K these formulae simplify to

$$f(x_n) \approx \frac{\varepsilon_n^{k}}{k'} f^{k}(\alpha) + \frac{\varepsilon_n^{k+1}}{(k+1)} f^{k+1}(\alpha)$$

$$f'(x_n) \approx \frac{\varepsilon_n^{k-1}}{(k-1)} f^{k}(\alpha) + \frac{\varepsilon_n^{k}}{k!} f^{k-1}(\alpha)$$

and so substituting in *

$$x_{n+1} = (\alpha + \varepsilon_n) - \frac{\lambda(\varepsilon_n + a\varepsilon_n^2}{k(1 + b\varepsilon_n)}$$

where $a = \dfrac{f^{k+1}(\alpha)}{(k+1)f^{k}(\alpha)}$ $\quad and \quad$ $b = \dfrac{f^{k+1}(\alpha)}{kf^{k}(\alpha)}$

Hence

$$x_{n+1} - \alpha = \varepsilon_n \left(1 - \frac{\lambda}{k}\right) - \frac{\lambda}{k}(a - b)\varepsilon_n^2$$

by choosing $\lambda = k$ we retain

quadratic convergence in this modified Newton Raphson and that

$$\varepsilon_{n+1} = x_{n+1} - \alpha \approx \frac{f^{k+1}(\alpha)}{k(k+1)f^{k}(\alpha)} e_n^2$$

we have therefore proved:

Theorem:

if $f(x)$ has a zero of multiplicity K at $x = \alpha$ the modified Newton Raphson formula

$$x_{n+1} = x_n - \frac{K^{f(x_n)}}{f'(x_n)}$$

has quadratic convergence.

Example:

Use the modified Newton Raphson method + find the double positive root of $x^4 - 4x^2 + 4 = 0$ starting at $x_1 = 1$

Solution:

In this case K = 2

so the formula is

$$x_{n+1} = x_n \frac{-2\left(x_n^4 - 4x_n^2 + 4\right)}{4x_n^3 - 8x_n}$$

or

$$x_{n+1} = \frac{x_n^4 - 4}{2x_n^3 - 4x_n}$$

n	x_n
1	1
2	1.5
3	1.4167
4	1.4142 Which is correct to 4

(ii) Stephenson's Method:

The N-R has the disadvantage that we have to workout $f'(x)$ Stephenson's produced a variant which avoids this but still converges quadratcally steffenson's iterative formula is

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f\left[x_n + f(x_n)\right] - f(x_n)}$$

which is easy to program and avoids having to write a procedure for $f`(x)$.

2. The General Iterative Method (Fixed Point Method):

The Newton-Raphson method is a particular example of a class of what are known as “iterative methods”. An iterative method is one in which an expression of the form.

$$x_{n+1} = F(x_n) \quad \ldots \qquad \ldots \qquad *$$

is used to produce the *(n+1)* st approximate (x_{n+1}) to the solution of the equation.

$$x = F(x) \qquad \ldots \qquad \ldots \qquad (1)$$

From the *n*th approximation (X_n).

Theorem:

If α is the exact solution of (1) so that $\alpha = F(\alpha)$ then * will converge to α from a sufficiently close starting value x_o if and only if $\left|F`(\alpha)\right| < 1$.

Proof: We are using * so that

$$x_{n+1} = F(x_n)$$

Suppose that $x_{n+1} = F(x_n)$ then

$$x_{n+1} = F(\alpha + \varepsilon_n) = F(\alpha) + \varepsilon_n F`(\alpha) + 0\left(\varepsilon_n^2\right)$$

$$= \alpha + \varepsilon_n F`(\alpha).$$

(Assuming that ε_n^2 may be ignored)

Thus the magnitude of the error at the $(n+1)$ st iteration is

$$\left|\varepsilon_n F`(\alpha)\right| < \left|\varepsilon_n\right| \;\; iff \;\; \left|F`(\alpha)\right| < 1$$

the errors $\left|\varepsilon_o\right|, \left|\varepsilon_1\right|, \; ...,$ will therefore form a decreasing sequence iff $\left|F`(\alpha)\right| < 1$ i.e. convergence will occur iff $\left|F`(\alpha)\right| < 1$

Definition:

If an iterative procedure for solving an equation converges to the solution in such a way that the errors $\varepsilon_n, \varepsilon_{n+1}$ at the *n*-th and *(n+1)* st iterations have a relationship of the form $\varepsilon_{n+1} = A\varepsilon_n^p$ then we say that the iterative procedure "converges with power P".

Corollary:

Since $|\varepsilon_{n+1}| = |\varepsilon_n||\varepsilon_n F`(\varepsilon)|$ the convergence is linear (i.e./ $\rho = 1$ ) unless $F`(\alpha) = 0$. The smaller the value of $|F`(\alpha)|$ the more rapid the convergence. Thus for fast convergence we should try to arrange* so that $|F`(\alpha)|$ is small.

Example:

Find the positive root of $x^2 - x - 1 = 0$ using iterative methods?

(Starting with $x = 1$)

Solution:

$$x_{n+1} = F(x_n)$$

(i) by writing the equation as

$$x = x^2 - 1$$

$$i.e / F(x) = x = x^2 - 1$$

$$F`(x) = 2x$$

$$F`(1) = 2 > 1$$

$$F'(2) = 4 > 1$$

Convergence will not occur since $\left|F`(\alpha)\right| > 1$.

(ii) by writing the equation as:

$$\left|F(x)\right| = x = 1 + \frac{1}{x}$$

$$F`(x) = \frac{-1}{x^2}$$

$$F`(1) = -1 < 1$$

$$F`(2) = \frac{-1}{4} < 1$$

convergence will occur since $\left|F`(\alpha)\right| < 1$.

Taking $x_o = 1$

n	x_n	$[x_{n+1} = F(x_n)]$
0	1	
1	2	
2	$3/2 \quad = \quad 1.5$	
3	$5/3 \quad = \quad 1.667$	
4	$8/5 \quad = \quad 1.600$	
5	$13/8 \quad = \quad 1.625$	
6	$21/13 \quad = \quad 1.615$	

Example:

Consider the equation $x^2 - e^x \cos(x) - 2.45 = 0$

Use three iteration of fixed point method with $x_o = 0.75$

To find the first positive nonzero root?

(ii) $$F(x) = x = \ln\left[\frac{2.45 - x^2}{\cos x}\right]$$

$$F`(x) = \frac{\cos x}{2.45 - x^2}\left[\frac{-2x\cos x + (2.45 - x^2)}{\cos^2 x}\right]$$

$$= \frac{-2x\cos x + (2.45 - x^2)\sin x}{(2.45 - x^2)\cos x}$$

$$= \frac{-2x}{(2.45 - x^2)} + \tan x$$

$$F`(1.5) = 0.8985 < 1$$

$$F`(0) = 0 < 1$$

it converges.

(it has root in the interval (0, 1.5).

Solution:

(i) $$F(x) = x = e^{x}\cos x - 2.45$$

$$F'(x) = \frac{\left[e^{x}\cos x - e^{x}\sin x\right] - e^{x}\cos x}{x^{2}} + \frac{2.45}{x^{2}}$$

$$F'(1.50) > 1$$

$$F'(0) > 1$$

convergence will not occur.

n	x_n	$x_{n+1} = F(x_n)$
0	0.75	
1	0.9477	
2	0.9781	
3	0.9833	