بسم الله الرحمن الرحيم

# Special Function

# (1):The Gamma-Function

We define the gamma-function $\Gamma(\nu)$

*By:*

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1} dx. \qquad (\nu > 0)..........(1)$$

Now integrating by parts

$$\int_0^\infty e^{-x} x^{\nu-1} dx = -\int_0^\infty x^{\nu-1} \frac{d}{dx}(e^{-x}) dx \qquad \frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$= -\left[x^{\nu-1}e^{-x}\right]_0^{\infty} + \int_0^{\infty} \frac{d}{dx}(x^{\nu-1})e^{-x}dx = (\nu-1)\int_0^{\infty} e^{-x}x^{\nu-2}dx.$$

$$\Gamma(\nu) = (\nu-1)\Gamma(\nu-1) \qquad (\nu > 1)$$ *

If integer $\nu = n,$ $\Gamma(n) = n(n-1)(n-2)..........1\Gamma(1)$

Also from equ(1)

$$\Gamma(1) = \int_0^{\infty} e^{-x}dx = 1 \qquad \left\{-e^{-\infty} + e^{0}\right\}$$

$\Gamma(n+1) = n!$ Integer. (Important)

The $\Gamma$ Function may therefore be thought of as a-generalization of the factorial function to which it reduces when $\nu$ .is position integer

The value of $\Gamma\left(\frac{1}{2}\right)$

eis often required and may b

obtained directly from the definition .

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx$$

by putting $x = u^2$ and integrating

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$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}\,.......(2)$$

( using line integral + multiple integral )

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy = \int_0^\infty dx \int_0^\infty e^{-\left(x^2+y^2\right)}\,dy$$

Polar coordinal = $\iint_{Rny} e^{-\left(x^2+y^2\right)}\,dxdy$

Using result (2) we may now obtain all other positive half-integral values from the recurrence relation (*)

e.g. $\Gamma\left(\tfrac{3}{2}\right)=\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$

$$\Gamma\left(\tfrac{7}{2}\right)=\frac{5}{2}\Gamma\left(\tfrac{5}{2}\right)=\frac{5}{2}.\frac{3}{2}\Gamma\left(\tfrac{3}{2}\right)=\frac{15}{4}.\frac{\sqrt{\pi}}{2}$$

The recurrence relation (*) is also useful in definition the $\Gamma$ -function for negative values of $\nu$ for re-writing (*) as:

$$\Gamma(\nu-1)=\frac{\Gamma(\nu)}{\nu-1} \qquad \text{..........(3)}$$

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e.g.

$$\Gamma\left(\frac{-3}{2}\right)=\frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}=\frac{4}{3}\sqrt{\pi}$$

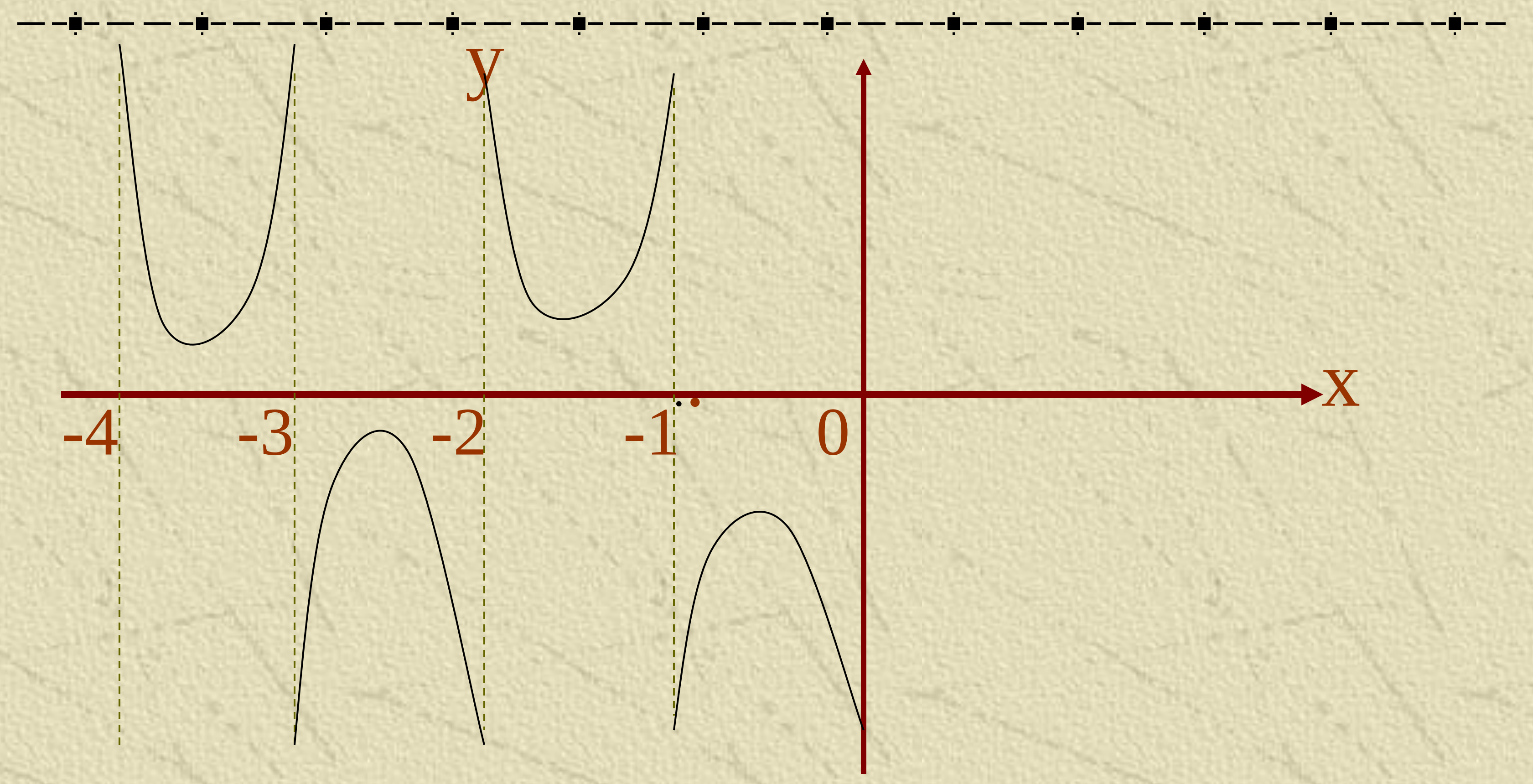
It is important to emphasize that the values of $\Gamma(\nu)$

for negative values of $\nu$ are not given by the

integral form (1)

but by the recurrence relation (3)

The graph of $\Gamma(\nu)$ for positive and negative values of $\nu$ is shown below:



## Example (1):

Evaluate $\Gamma(5)$

: Solution:

$$\Gamma(n+1) = n! \qquad \therefore \Gamma(5) = 4! = 4*3*2*1 = 24$$

## Example (2):

Evaluate $\Gamma(3.5)$

..

Solution :

$$\Gamma(\nu) = (\nu - 1)\Gamma(\nu - 1)$$

$$\Gamma(3.5) = 2.5\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.88623)$$

where $\Gamma(1.5)$ can be found from Gamma tables .

Example(3):

Evaluate the integral $I = \int_0^\infty t^{5.2} e^{-t^2} dt$

Solution:

Put $t^2 = x$

$$\therefore I = \frac{1}{2}\int_0^\infty x^{2.1} e^{-x} dx = \frac{1}{2}\Gamma(3.1) = \frac{1}{2}(2.1)\Gamma(2.1) = \frac{(2.1)(1.1)}{2}\Gamma(1.1)$$

$$= \frac{1}{20}(2.1)(1.1)\Gamma(0.1)$$

from tables: $\Gamma(0.1) = 9.5135$

then: $I = 1.10$ (2d.p.)

## Example (4):

Evaluate:

$$I = \int_0^1 \sqrt{\ln\left(\frac{1}{t}\right)}\,dt$$

Solution:

Let $t = e^{-x}$ ..

Then: $$I = \int_0^\infty x^{1/2} e^{-x}\,dx = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Example (5):

$$I = \text{If} \int_0^{\pi/2} \left(\tan^3\theta + \tan^5\theta\right) e^{-\tan^2\theta}\, d\theta$$

Evaluate I.

Solution:

Let $\tan^2\theta = x$

Then:

$$I = \tfrac{1}{2}\int_0^{\infty} x e^{-x}\, dx = \frac{1}{2}\Gamma(2) = \frac{1}{2}$$

Example (6):

If $I = \int_0^1 \left(x \ln x\right)^3 dx$ Evaluate. . .

Solution:

Let $-t = \ln x$

$$\therefore e^{-t} = x$$

$$dx = -e^{-t} dt$$

$$\therefore \int_0^1 (x \ln x)^3 \, dx = \int_\infty^0 \left(e^{-t}\right)^3 (-t)^3 \left(-e^{-t}\right) dt$$

$$\mathrm{I} = -\int_0^\infty + t^3 e^{-t} e^{-3t} \, dt = -\int_0^\infty t^3 e^{-4t} \, dt$$

$let$

$u = 4t$

$$\frac{1}{4} du = dt$$

$$I = -\frac{1}{4}\int_0^{\infty}\left(u/4\right)^3 e^{-u}\,du = -\frac{1}{256}\int_0^{\infty} u^3 e^{-u}\,du$$

$$\Gamma(4) = 3\Gamma(3) = 3!$$

$$\therefore I = -\frac{1}{256} * 3 * 2 * 1 = \frac{3}{128}$$

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# Alternative Forms of Gamma-Function:

1- $\chi = \nu^2 \qquad dx = 2\nu d\nu$

$$\therefore \Gamma(\nu) = \int_0^\infty e^{-\chi}\chi^{\nu-1}d\chi = \int_0^\infty e^{-\nu^2}\nu^{2(\nu-1)}2\nu d\nu = 2\int_0^\infty e^{-\nu^2}\nu^{2\nu-1}d\nu$$

2-An alternative definition of $\Gamma(\nu)$ due to Euler is :

$$\Gamma(\nu) = \lim_{n\to\infty}\left\{\frac{n!*n^\nu}{\nu(\nu+1)........(\nu+n)}\right\}.$$

This form is valid for positive and negative $\nu$ and shows clearly the singularities of $\Gamma(\nu)$ at $\nu = 0,-1,-2,.......$ and so on .

# *Homework*

1) If $\Gamma(1.1) = 0.951$ find $\Gamma(4.1)$ *and* $\Gamma(-3.9)$

2) Evaluate $I = \int_1^\infty e^{2\chi - \chi^2}\, dx$

3) Evaluate $I = \int_1^1 \chi^2 \left(\ln \frac{1}{x}\right)^3 dx$

4) Evaluate $I = \int_0^\infty t^4 e^{-2t^3}\, dt$

5) Evaluate $I = \int_0^\infty \frac{u^2}{(1+u)^5}\, du$ Hint $\left(v = \frac{u}{1+u}\right)$

# Answer

5)Let $v = \frac{u}{1+u}$ $\therefore \frac{dv}{du} = \frac{1}{(1+u)^2}$ $dv = \frac{1}{(1+u)^2} du$

$$\therefore \int_0^\infty v^2 (1-v) dv.$$

Let $v = \sin^2\theta$ $dv = 2\sin\theta\cos\theta d\theta$

$$\therefore \mathrm{I} = 2\int_0^{\pi/2} \sin^4\theta (1-\sin^2\theta)\sin\theta\cos\theta \quad d\theta = 2\int_0^{\pi/2} \sin^5\theta\cos^3\theta \quad d\theta = \beta(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{1}{12}$$

# (2):The Beta-Function

We define the Beta –Function $\beta(p,q)$

by the integral:

$$\beta(p,q)=\int_0^1 \chi^{p-1}(1-x)^{q-1}dx \quad (p>0, q>0)\ldots\ldots\ldots(1)$$

Suppose $u=1-x$ then $x=1-u$ $\therefore dx=-du$

$$\beta(p,q)= -\int_1^0 (1-u)^{p-1}(u)^{q-1}du$$

$$=\int_0^1 u^{q-1}(1-u)^{p-1}du=\beta(q,p)$$

$$\beta(p,q)=\beta(q,p).$$

Going back to the definition:

$$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

Integrating by parts by assuming

$$u=(1-x)^{q-1} \qquad , \frac{dv}{dx}=x^{p-1}$$

Then

$$\beta(p,q)=\left[\frac{x^p}{p}(1-x)^{q-1}\right]_0^1+\frac{(q-1)}{p}\int_0^1 x^p(1-x)^{q-2}\,dx$$

$= 0$

Then

$$\int_0^1 x^p(1-x)^{q-2}\,dx=\int_0^1 x^{p-1}[1-(1-x)](1-x)^{q-2}\,dx$$

$$=\int_0^1 x^{p-1}(1-x)^{q-2}\,dx-\int_0^1 x^{p-1}(1-x)^{q-1}\,dx.$$

$$\therefore \beta(p,q) = \left(\frac{q-1}{p}\right)\{\beta(p,q-1) - \beta(p,q)\}$$

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$$\beta(p,q).\left(1+\frac{q-1}{p}\right) = \left(\frac{q-1}{p}\right)\beta(p,q-1)$$

$$\boxed{\therefore \beta(p,q) = \left(\frac{q-1}{p+q-1}\right)\beta(p,q-1)} \qquad *$$

$$(p+q-1)\beta(p,q) = (q-1)\beta(p,q-1)$$

$$(p+q-1)\beta(p,q) = (p-1)\beta(p-1,q)$$

This comes from the symmetric relation

$$\beta(p,q) = \beta(q,p)$$

$$\therefore (p+q-1)\beta(p,q) = (p-1)\left\{\frac{(q-1)}{(p+q-2)}\beta(p-1,q-1)\right\}$$

See *

$$\therefore \beta(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)}\beta(p-1,q-1)$$

An alternative form of the Beta-Function , obtained

from (1) by putting $\chi = \sin^2\theta$ is:

$$\beta(p,q) = \int_0^{\pi/2} \sin^{2p-2}\theta \cos^{2q-2}\theta \, 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2q-1}\theta \, d\theta \qquad (2)$$

# Relation of Beta-Function to Gamma-Function

$$\Gamma(\nu) = \int_0^\infty e^{-x} x^{\nu-1} dx = 2\int_0^\infty e^{-t^2} t^{2\nu-1} dt \qquad (x = t^2)$$

$$\therefore \Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-x^2} x^{2p-1} dx \int_0^\infty e^{-y^2} y^{2q-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2p-1} y^{2q-1} dx dy$$

Transform integral to polar coordinates

$$x = r\cos\theta \qquad 0 \le r \le \infty$$

$$y = r\sin\theta \qquad 0 \le \theta \le \pi/2$$

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$$\therefore \Gamma(p)\Gamma(q) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2p-1}\cos^{2p-1}\theta\, r^{2q-1}\sin^{2q-1}\theta\, r\,dr\,d\theta$$

Using Jacobean ..

$$J = \frac{\partial}{\partial}\left(\frac{x, y}{r, \theta}\right) dr\,d\theta = r\,dr\,d\theta$$

$$J = \frac{\partial x}{\partial r}\cdot\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\cdot\frac{\partial y}{\partial r}$$

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty e^{-r}\ r^{2p+2q-1}dr \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta d\theta$$

$$= \Gamma(p+q)\beta(q,p) = \Gamma(p+q)\beta(p,q)$$

$$\therefore\ \beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 \qquad \Gamma(1) = 1$$

$$= 2\int_0^{\pi/2} \sin^{1-1}\theta \cos^{1-1}\theta d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \pm\sqrt{\pi}$$

But

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = +\sqrt{\pi}$$

Useful alternative forms of Beta-Function:

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

Change variables $v = \dfrac{x}{1-x}$

Limits

$x \to 0 \qquad v \to 0$

$x \to 1 \qquad v \to \infty$

$$x = \frac{\nu}{1+\nu} \qquad dx = \left( \frac{1}{1+\nu} - \frac{\nu}{(1+\nu)^2} \right) d\nu = \frac{1}{(1+\nu)^2} d\nu$$

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$$\therefore \beta(p,q) = \int_0^\infty \left( \frac{\nu}{1+\nu} \right)^{p-1} \left( \frac{1}{1+\nu} \right)^{q-1} \frac{1}{(1+\nu)^2} d\nu$$

$$\beta(p,q) = \int_0^\infty \frac{\nu^{p-1}}{(1+\nu)^{p+q}} d\nu$$

We may assume

$$\beta(p,q) = \int_0^1 \frac{\nu^{p-1}}{(1+\nu)^{p+q}} d\nu + \int_0^\infty \frac{\nu^{p-1}}{(1+\nu)^{p+q}} d\nu$$

In the 2nd integral put $v = \frac{1}{w}$

$$\therefore \int_1^\infty \frac{v^{p-1}}{(1+v)^{p+q}}\,dv = \int_0^1 \frac{1}{w^{p-1}} \frac{1}{\left(1+\frac{1}{w}\right)^{p+q}} \frac{-1}{w^2}\,dw = \int_0^1 \frac{w^{p+q-p+1-2}}{(1+w)^{p+q}}\,dw$$

$$= \int_0^1 \frac{w^{q-1}}{(1+w)^{p+q}}\,dw = \int_0^1 \frac{v^{q-1}}{(1+v)^{p+q}}\,dv.$$

$$\therefore \beta(p,q) = \int_0^1 \frac{v^{p-1} + v^{q-1}}{(1+v)^{p+q}}\,dv$$

## Example:

Evaluate $\mathrm{I} = \int_0^1 \frac{dx}{\sqrt{1 - x^4}}$

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:Solution: let $x^4 = u$

$$\therefore \mathrm{I} = \frac{1}{4}\int_0^1 \frac{u^{-3/4}}{\sqrt{1-u}}\,du = \frac{1}{4}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}.$$

## Example:

Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$

Solution: Let $u = \sin^2\theta$

$$\therefore \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \frac{1}{2}\int_0^1 \frac{u^{-1/4}}{\sqrt{1-u}}\, du = \frac{1}{2}\beta\left(3/4, 1/2\right) = \frac{\Gamma\left(3/4\right)\Gamma\left(1/2\right)}{2\Gamma\left(5/4\right)}$$

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Since

$$\Gamma\left(5/4\right) = \frac{1}{4}\Gamma\left(1/4\right) \quad \text{and} \quad \Gamma\left(1/2\right) = \sqrt{\pi}$$

$\therefore$

Then

$$\int_0^{\pi/2} \sin\theta\, d\theta = 2\sqrt{\pi}\,\frac{\Gamma\left(3/4\right)}{\Gamma\left(1/4\right)} = 1.198$$

# Example:

Evaluate $$\mathrm{I} = \int_0^{\infty} \frac{x^{3/2}}{(1+x)^{7/2}}\,dx$$

# Solution:

$$\beta(p,q) = \int_0^{\infty} \frac{\nu^{p-1}}{(1+\nu)^{p+q}}\,d\nu = \beta\left(5/2, 1\right) = \frac{\Gamma(2.5)\Gamma(1)}{\Gamma(3.5)}$$

$$= \frac{\Gamma(2.5).1}{2.5\Gamma(2.5)} = \frac{1}{2.5} = 0.4$$

# Example:

Evaluate $$I = \int_0^{1/4} \sqrt{\frac{(1-4x)^5}{x}}\, dx$$

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Solution: $$I = \int_0^{1/4} x^{-1/2} (1-4x)^{5/2}\, dx$$

Let $u = 4x$ $\qquad \therefore \frac{1}{4}\, du = dx$

$$\therefore I = \frac{1}{4}\int_0^1 \left(\frac{1}{4}\right)^{-1/2} u^{-1/2} (1-u)^{5/2}\, du = \frac{1}{2}\int_0^1 u^{-1/2}(1-u)^{5/2}\, du = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{7}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(4)} = \frac{\sqrt{\pi}}{2}\,\frac{\Gamma(3.5)}{\Gamma(4)}$$

$$= \frac{\sqrt{\pi}}{2} \frac{2.5\Gamma(2.5)}{3\Gamma(3)} = \frac{\sqrt{\pi}}{2} \frac{(2.5)(1.5)\Gamma(1.5)}{(3)(2)\Gamma(2)} = \frac{\sqrt{\pi}}{2} \frac{(2.5)(1.5)(1)\Gamma\left(\frac{1}{2}\right)}{(3)(2)(1)\Gamma(1)}$$

$$= \frac{\sqrt{\pi}}{2} \frac{(2.5)(1.5)(1)\sqrt{\pi}}{(3)(2)(1)} = \frac{\sqrt{\pi}}{12} (2.5)(1.5)$$

# (3):The Error-Function

The error function erf $x$ is defined as

$$erf x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

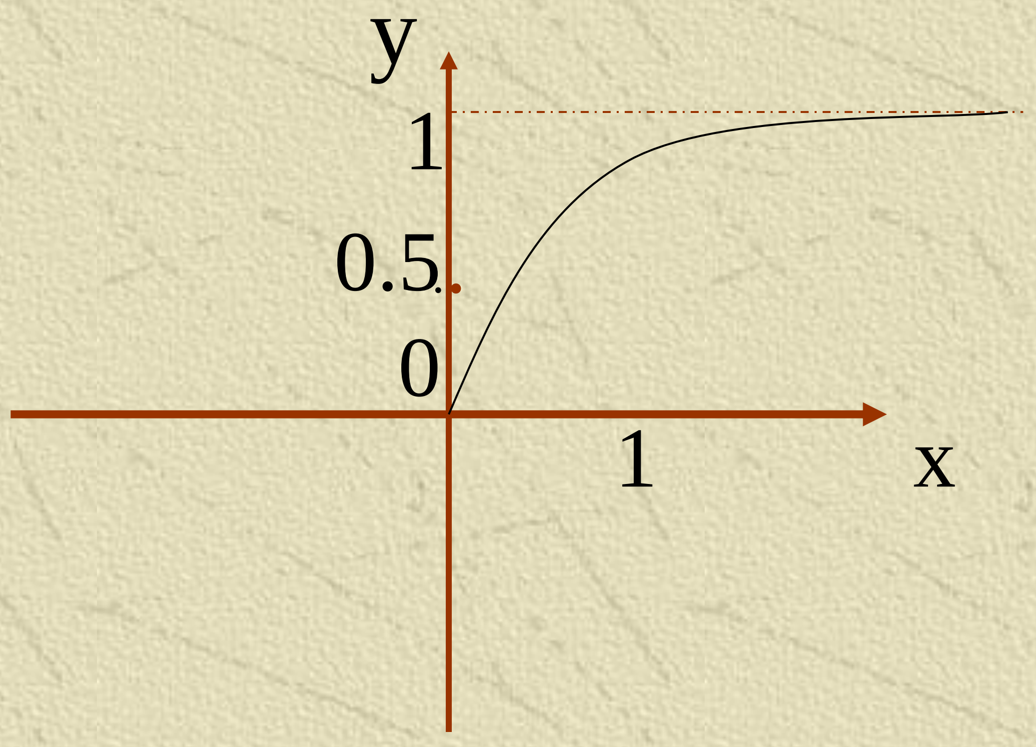
and clearly represents (apart from the factor $2/\sqrt{\pi}$) the area under the curve $e^{-u^2}$ from $u = 0$ to $u = x$

We see immediately that $erf(0) = 0$ and that $erf(\infty) = 1$ Since $\int_0^\infty e^{-u^2} du = \frac{\sqrt{\pi}}{2}$ $\quad v = u^2$

$$\therefore \int_0^\infty e^{-u^2} du = \frac{1}{2}\int_0^\infty v^{-1/2} e^{-v} dv = \frac{1}{2}\Gamma\left(1/2\right) = \frac{\sqrt{\pi}}{2}$$

$$\therefore erf(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du = \frac{2}{\sqrt{\pi}} . \frac{\sqrt{\pi}}{2} = 1$$

# The graph of the error function is as shown below



# Solution of ordinary Differential Equations

2nd order $\frac{d^2 z}{dt^2} + p(t)\frac{dz}{dt} + q(t)z = 0$

# Power Series

Suppose $t$ is independent variable, to some

particular value of $t$ then:

$$a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + a_3(t-t_0)^3 + .......... = \sum_{n=0}^{\infty} a_n (t-t_0)^n.$$

This is a power series centered about $t = t_0$

The sum of terms to $N$ is

$$S_N(t) = \sum_{n=0}^{N} a_n (t-t_0)^n, .....R_N(t) = \sum_{n=N+1}^{\infty} a_n (t-t_0)^n$$

Where $S_N(t) = N^{th}$ Partial sum.

$R_N(t) =$ Remainder

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Suppose that when $t = t_1$ then sequence

$S_1(t_1), S_2(t_1), S_3(t_1)\ldots\ldots \to$ to a definite

Limit, the series is convergent(or converges) at

$$t = t_1$$

e.g. /

$$S_N(t_1) = \sum_{n=0}^{\infty} a_n (t_1 - t_0)^n = \lim_{N\to\infty} S_N(t_1) = S(t_1).$$

When the series does not tend to a definite limit .then the series is divergent

For some positive numbers $R$:

Series converges $\forall t$ such that $|t - t_0| < R$

Diverges $\forall t$ such that $|t - t_0| > R$

$R$ is called the interval or radius of convergence.

(I) $\dfrac{1}{R} = \lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$ (if the limit exists)

(II) $\dfrac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ (if the limit exists ) e.g. / $$1+t+t^2+t^3+\ldots\ldots\ldots\ldots=\sum_{n=0}^{\infty} t^n \qquad t_0=0 \qquad a_n=1\,\forall n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1=\frac{1}{R}$$

$$\lim_{n\to\infty}|a_n|=1=\frac{1}{R}$$

$$\therefore R=1$$

Series converges for $\because\ |t-t_0|<R \qquad |t|<1$

e.g. / $$S_N(t)=1+t+t^2+\ldots\ldots\ldots+t^N$$

$$tS_N(t)=t+t^2+\ldots\ldots\ldots+t^N+t^{N+1}\,sub$$

$$(1-t)S_N(t)=1-t^{N+1}$$

$$S_N(t)=\frac{1-t^{N+1}}{(1-t)}$$

---

$$\therefore \lim_{N\to\infty} S_N(t)=\lim_{N\to\infty}\left\{\frac{1-t^{N+1}}{1-t}\right\}=\frac{1}{1-t}=S_\infty(t)$$ Where $|t|<1$

But if $t\to\infty$ then $series \to\infty$

$\therefore$ Limit does not exist

e.g. / $$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\dots\dots\dots=\sum_{n=0}^{\infty}\frac{t^n}{n!}$$

$$t_0=0$$

$$a_n=1/n!$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 = \frac{1}{R}$$

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$$\therefore R = \infty \qquad \lim_{N\to\infty} S_N(t) = S(t) = e^t.$$

e.g. /

$$1 + (t-1) + 2!(t-1)^2 + 3!(t-1)^3 + \ldots\ldots\ldots = \sum_{n=0}^{\infty} n!(t-1)^n.$$

$$t_0 = 1, a_n = n!$$

$$\left|\frac{a_{n+1}}{a_n}\right| = n+1 \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty = \frac{1}{R} \qquad R = 0$$

The series converges if $|t-1|<0$

A function can be written in terms of power series for certain ranges if value of $t$:

$$f(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+\ldots\ldots\ldots=\sum_{n=0}^{\infty}a_n(t-t_0)^n.$$

$$|t-t_0|<R$$

and if it is possible to write $f(t)$ in power series

then $f(t)$ is analytic at $t=t_0$

Taylor series of $f(t)$ about $t=t_0$

$$f(t)=\sum_{n=0}^{\infty}a_n(t-t_0)^n.$$

$$\frac{d^r f(t)}{dt^r} = \sum_{n=0}^{\infty} a_n (n)(n-1)\ldots\ldots(n-r+1)(t-t_0)^{n-r}$$

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$$\frac{d^r f(t)}{dt^r}\Big|_{t=t_0} = \sum_{n=0}^{\infty} a_n (n)(n-1)(n-2)\ldots\ldots(n-r+1)(t_0-t_0)^{n-r}$$

$$= a_r r(r-1)(r-2)\ldots\ldots\ldots(1) = r!a_r$$

$$a_n = \frac{1}{n!}\frac{d^n f(t)}{dt^n}\Big|_{t=t_0}$$

## To Find the Radius:

$$f(t) = \sum_{n=0}^{\infty} a_n (t-t_0)^n.$$

(i) Find the value of $t$ at which $f$ goes wrong

i.e. / $f$ is infinite.

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(ii)For general several such values, usually complex

$$t_1 + it_2 \qquad \sqrt{(t_1 - t_0)^2 - t_2^2}$$

( The smallest one of these roots gives the value of R)

..

$$a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + \ldots.. + a_N(t - t_0)^N$$

It is called a polynomial series (not power series)

Suppose we want to find the solution of equation

$$\frac{d^2z}{dt^2} + p(t)\frac{dz}{dt} + q(t)z = 0$$ in terms of power series:

Case (1):

Both $p(t)$ and $q(t)$ have convergent Taylor expansion centered at $t = t_0$

(both are analytic at $t = t_0$).

$t_0$ – ordinary point of the equation.

The general solution of the equation can be written as:

$$z(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

(Series solution of H.2nd order.diff.equation)

## Case (2):

$p(t)$ or $q(t)$ [or both] are not analytic at $t = t_0$

---

Then $t = t_0$ is known as singular point of the equation and it is not possible to write the general solution but if $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are both analytic at $t = t_0$

Then $t = t_0$ is regular singular point of the differential equation we can then find at least one solution of the form

$$z(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^{n+c}$$

Where C – constant and where $a_0 \neq 0$

(Singular point method of Frobenius).

Example:

$$\frac{d^2z}{dt^2} - 2t\frac{dz}{dt} - 2z = 0$$

$$z(0) = 3$$

$$z'(0) = 5$$

Solution:

$$p(t) = -2t$$

$$q(t) = -2$$

$$t = t_o$$

$$t_o = 0$$

Analytic since $p(t)$، $q(t)$ do not go to $\infty$

when putting $t = t_0 = 0$ where $p(t) = 0$, $q(t) = -2$

$$\therefore z = \sum_{n=0}^{\infty} a_n t^n$$

$$\frac{dz}{dt} = \sum_{n=0}^{\infty} n a_n t^{n-1}$$

$$\frac{d^2 z}{dt^2} = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

$$\therefore \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2=N} - 2\sum_{n=0}^{\infty} n a_n t^n - 2\sum_{n=0}^{\infty} a_n t^n = 0$$

$$\sum_{N=0}^{\infty}\{(N+1)(N+2)a_{N+2} - 2Na_N - 2a_N\}t^N = 0$$

---

$$\frac{a_{N+2}}{a_N} = \frac{2(N+1)}{(N+1)(N+2)} = \frac{2}{N+2}$$ For $N = 0,1,2,\ldots\ldots.$

$$\therefore z(t) = \sum_{N=0}^{\infty} a_N t^N = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots\ldots\ldots$$

$$= a_0\left[1 + \frac{a_2}{a_0}t^2 + \frac{a_4}{a_2}.\frac{a_2}{a_0}t^4 + \frac{a_6}{a_4}.\frac{a_4}{a_2}.\frac{a_2}{a_0}t^6 + \ldots\ldots..\right]$$

$$+ a_1\left[t + \frac{a_3}{a_1}t^3 + \frac{a_5}{a_3}.\frac{a_3}{a_1}t^5 + \frac{a_7}{a_5}.\frac{a_5}{a_3}.\frac{a_3}{a_1}t^7 + \ldots\ldots..\right]$$

$$= a_0 \left[ 1 + (1)t^2 + \left( \tfrac{1}{2} \right)(1)t^4 + \left( \tfrac{1}{3} \right)\left( \tfrac{1}{2} \right)(1)t^6 + \ldots\ldots. \right]$$

$$+ a_1 \left[ t + \left( \tfrac{2}{3} \right)t^3 + \left( \tfrac{2}{5} \right)\left( \tfrac{2}{3} \right)t^5 + \left( \tfrac{2}{7} \right)\left( \tfrac{2}{5} \right)\left( \tfrac{2}{3} \right)t^7 + \ldots\ldots\ldots \right]$$

$$\therefore z(t) = a_0 z_1(t) + a_1 z_2(t)$$ Where

$$z_1(t) = 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \ldots\ldots\ldots = e^{t^2}$$

$$z_2(t) = t + \frac{2}{3}t^3 + \frac{4}{15}t^5 + \frac{8}{105}t^7 + \ldots\ldots.. +$$

Two linear independent solution of the equation.

Now given $z(0) = 3$

$$\therefore z(0) = a_0(1) + a_1(0) = 3$$

i.e. / $a_0 = 3$

$$z'(0) = 5 = a_0(0) + a_1(1)$$

i.e / . $a_1 = 5$

$$\therefore z(t) = 3z_1(t) + 5z_2(t)$$

<u>Example:</u>

$$t^2 \frac{d^2 z}{dt^2} + t\frac{dz}{dt} + t^2 z = 0$$

(Called Bessel's equation of order zero)

To find solution, power series centered about $t=0$

$p(t)=\frac{t}{t^2}=\frac{1}{t}$ at $t=0$ $\quad p(t)\to\infty$ $\quad\therefore$ not analytic

---

$q(t)=\frac{t^2}{t^2}=1$ at $t=0$ $\quad q(t)=1$ $\quad\therefore$ analytic.

Because $p(t)$ is not analytic we write

$$\left.\begin{matrix}(t-t_0)p(t)\\(t-t_0)^2q(t)\end{matrix}\right]$$ are analytic at $t=t_0$

Now, we assume solution of the form

$$z = \sum_{n=0}^{\infty} a_n t^{n+c}$$

$$\frac{dz}{dt} = \sum_{n=0}^{\infty} (n+c) a_n t^{n+c-1}$$

$$\frac{d^2 z}{dt^2} = \sum_{n=0}^{\infty} (n+c)(n+c-1) a_n t^{n+c-2}$$

$$\therefore \sum_{n=0}^{\infty} (n+c)(n+c-1) a_n t^{n+c} + \sum_{n=0}^{\infty} (n+c) a_n t^{n+c} + \sum_{n=0}^{\infty} a_n t^{n+c+2} = 0$$

Coefficient of $t^N$

$$\sum_{N=0}^{\infty} \{ N(N-1) a_{N-c} + N a_{N-c} + a_{N-c-2} \} t^N = 0$$

$$\therefore N^2 a_{N-c} + a_{N-c-2} = 0$$

At $N = c$

$$c^2 a_0 + a_{-2} = 0$$

$$\therefore c^2 = 0$$

$$\therefore c = 0$$

There is only one value of c , therefore we can find only

one solution:

$$\therefore \sum_{n=0}^{\infty} (n)(n-1) a_n t^n + \sum_{n=0}^{\infty} (n) a_n t^n + \sum_{n=0}^{\infty} a_n t^{n+2} = 0$$

At $N = c + 1 = 1$

$$a_{-1} + a_1 = 0$$

$$\therefore a_1 = a_3 = a_5 = \ldots\ldots = 0$$

---

At $N = c + 2 = 2$

$$4a_2 + a_0 = 0$$

$$\frac{a_2}{a_0} = -\frac{1}{4}$$

At $N = c + 4 = 4$

$$16a_4 + a_2 = 0$$

$$\frac{a_4}{a_2} = -\frac{1}{16}$$

$$z(t)=a_0\left[1+\frac{a_2}{a_0}t^2+\frac{a_4}{a_2}.\frac{a_2}{a_0}t^4+......\right]=a_0\left[1+\left(\frac{-1}{4}\right)t^2+\left(\frac{-1}{16}\right)\left(\frac{-1}{4}\right)t^4+.....\right]=a_0 g(t)$$

Where $g(t)$ is Bessel's function of order zero .

$$g(t)=z_1(t)$$

Now ‘ $\dfrac{d^2z}{dt^2}+p(t)\dfrac{dz}{dt}+q(t)z=0$

..

$$w(t)=\int p(t)\quad dt,\quad z_1(t):$$

$$\therefore z_2(t)=z_1(t)\int\frac{e^{-w(t)}}{\left[z_1(t)\right]^2}dt$$

$$p(t)=\frac{1}{t} \qquad \therefore w(t)=\ln t$$

$$e^{-w(t)}=e^{-\ln t}=\frac{1}{t}$$

$$\therefore z_2(t)=z_1(t)\int\frac{dt}{t[z_1(t)]^2}$$

Called Neumann function.

Example:

$$\left(t^2-2t\right)\frac{d^2z}{dt^2}+5(t-1)\frac{dz}{dt}+3z=0$$

$$z(1)=7$$

$$z'(1)=3$$

Solution:

$$\frac{Q(t)}{p(t)} = \frac{5(t-1)}{t^2 - 2t} \to \infty \quad t = 0,2$$

$$\frac{R(t)}{p(t)} = \frac{3}{t^2 - 2t} \to \infty \quad t = 0,2$$

Not analytic at $t_0 = 0,2$

At $t_0 = 1$ both are analytic.

Assume a solution of the form

$$z = \sum_{n=0}^{\infty} a_n (t - t_0)^n = \sum_{n=0}^{\infty} a_n (t-1)^n$$

$$\therefore z' = na_n(t-1)^{n-1}$$

---

$$z'' = n(n-1)a_n(t-1)^{n-2}$$

$$\therefore \sum (t^2-2t)(n)(n-1)a_n(t-1)^{n-2} + 5(t-1)na_n(t-1)^{n-1} + 3a_n(t-1)^n = 0$$

$$\sum_{n=0}^{\infty}\{(t-1)^2-1\}(n)(n-1)a_n(t-1)^{n-2} + \sum_{n=0}^{\infty}5(t-1)na_n(t-1)^{n-1} + \sum_{n=0}^{\infty}3a_n(t-1)^n = 0$$

$$\sum_{n=0}^{\infty}n(n-1)a_n\{(t-1)^n-(t-1)^{n-2}\} + \sum_{n=0}^{\infty}5na_n(t-1)^n + \sum_{n=0}^{\infty}3a_n(t-1)^n = 0$$

Coefficient of $(t-1)^N$

$$\sum_{N=0}^{\infty}N(N-1)a_N(t-1)^N - \sum_{N=0}^{\infty}(N+2)(N+1)a_{N+2}(t-1)^N + \sum_{N=0}^{\infty}5Na_N(N-1)^N$$

$$+\sum_{N=0}^{\infty}3a_N(t-1)^N = 0$$

$$\sum_{N=0}^{\infty}\{N(N-1)a_N-(N+2)(N+1)a_{N+2}+5Na_N+3a_N\}(t-1)^N=0$$

$$\therefore -(N+2)(N+1)a_{N+2}+\{N(N-1)+5N+3\}a_N=0$$

$$-(N+2)(N+1)a_{N+2}+(N^2+4N+3)a_N=0$$

$$-(N+2)(N+1)a_{N+2}+(N+1)(N+3)a_N=0$$

$$\therefore \frac{a_{N+2}}{a_N}=\frac{(N+1)(N+3)}{(N+1)(N+2)}=\frac{N+3}{N+2}$$

$$z(t)=a_0\left\{1+\frac{a_2}{a_0}(t-1)^2+\frac{a_4}{a_2}.\frac{a_2}{a_0}(t-1)^4+......\right\}+a_1\left\{(t-1)+\frac{a_3}{a_1}(t-1)^3+.....\right\}$$

$$=a_0\left\{1+\frac{3}{2}(t-1)^2+\frac{15}{8}(t-1)^4+.....\right\}+a_1\left\{(t-1)+\frac{4}{3}(t-1)^3+......\right\}$$

But $z(1) = a_0 = 7$

$$z'(1) = a_1 = 3$$

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

$$\therefore z(t) = 7\left\{1 + \frac{3}{2}(t-1)^2 + ....\right\} + 3\left\{(t-1) + \frac{4}{3}(t-1)^3 + .....\right\}$$

For all t satisfy $|t-1| < 1$

***********************************

# Legendre's Equation

The equation $\left(1-x^2\right)\dfrac{d^2z}{dx^2}-2x\dfrac{dz}{dx}+\ell(\ell+1)z=0$

when $\ell$ is a real constant , is Legendre's equation and

may be solved by applying the Frobenius method.

Let $z=\sum_{n=0}^{\infty}a_n x^n$

$$\frac{dz}{dt}=\sum_{n=0}^{\infty}na_n x^{n-1}$$

$$\frac{d^2z}{dx^2}=\sum_{n=0}^{\infty}n(n-1)a_n x^{n-2}$$

$$\therefore \sum_{n=0}^{\infty}(1-x^2)n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 2xna_n x^{n-1} + \sum_{n=0}^{\infty} \ell(\ell+1)a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n)(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \ell(\ell+1)a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n)(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \ell(\ell+1)a_n x^n = 0$$

Coefficient of $x^N$

$$\sum_{n=0}^{\infty}\{(N+2)(N+1)a_{N+2} - N(N-1)a_N - 2Na_N + \ell(\ell+1)a_N\}x^N = 0$$

$$\therefore (N+2)(N+1)a_{N+2} - \{N^2 - N + 2N - \ell(\ell+1)\}a_N = 0$$

$$\frac{a_{N+2}}{a_N} = \frac{N^2 + N - \ell(\ell+1)}{(N+2)(N+1)}$$

$$\therefore z(x) = \sum_{N=0}^{\infty} a_N x^N$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .......$$

$$= a_0 \left\{ 1 + \frac{a_2}{a_0} x^2 + \frac{a_4}{a_2} . \frac{a_2}{a_0} x^4 + ...... \right\} + a_1 \left\{ x + \frac{a_3}{a_1} x^3 + \frac{a_5}{a_3} . \frac{a_3}{a_1} x^5 + ...... \right\}$$

$$= a_0 \left\{ 1 - \frac{\ell(\ell+1)}{2!} x^2 + \frac{\ell(\ell-2)(\ell+1)(\ell+3)}{4!} x^4 - ........... \right\}$$

$$+ a_1 \left\{ x - \frac{(\ell-1)(\ell+2)}{3!} x^3 + \frac{(\ell-1)(\ell-3)(\ell+2)(\ell+4)}{5!} x^5 + ........ \right\} .........(1)$$

Where $a_0$ and $a_1$ are arbitrary constants.

When $\ell = n$ where n is an integer, either one or the other of these two series terminates

i.e. / $z_1(x)$ or $z_2(x)$

e.g / if $\ell = 2$ then all terms in

$z_1(x)$ beyond $x^2$ .are zero

Similarly if $\ell = 3$ all terms in $z_2(x)$ beyond $x^3$ .are zero

The resulting polynomials in x denoted by $p_n(x)$ are called

legendre polynomials( $a_0$. and $a_1$ being chosen so that

each polynomial has the value unity when $x = 1$)

the first few of these polynomials are:

$$p_0(x) = 1$$

$$p_1(x) = x$$

---

$$p_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

..

Non-terminating series is of no interest here.

A straight forward rewriting by means of factorials enable us to express the solution (1) in the form

$$z = a_0 \sum_k (-)^k \frac{\left(\frac{\ell}{2}\right)!}{\left(\frac{\ell}{2}+k\right)!} \frac{\left(\frac{\ell}{2}\right)!}{\left(\frac{\ell}{2}-k\right)!} \frac{(\ell+2k)!}{\ell!} \frac{x^{2k}}{(2k)!}$$

$$+ a_1 \sum_k (-)^k \frac{\left(\frac{\ell-1}{2}\right)!}{\left(\frac{\ell-1}{2}+k\right)!} \frac{\left(\frac{\ell-1}{2}\right)!}{\left(\frac{\ell-1}{2}-k\right)!} \frac{(\ell+2k)!}{\ell!} \frac{x^{2k+1}}{(2k+1)!}$$

If $\ell = 0,2,4,\ldots\ldots$ .the first series becomes a polynomial

We shall introduce the new summation index

$$r = \frac{\ell}{2} - k$$ and write this solution as

$$a_0 \frac{\left[\left(\frac{\ell}{2}\right)!\right]^2}{\ell!} (-)^{\ell/2} \sum_r (-)^r \frac{(2\ell - 2r)!}{(\ell - r)!\,r!} \frac{x^{\ell - 2r}}{(\ell - 2r)!}$$

If on the other hand $\ell = 1,3,5,\ldots\ldots$ it is the second series

concerns us.

In this series we define $r = \left[\frac{(\ell-1)}{2}\right] - k$ and obtain the solution.

$$\frac{a_1}{2}\frac{\left[\left(\frac{\ell-1}{2}\right)!\right]^2}{\ell!}(-)^{\frac{\ell-1}{2}}\sum_{r}(-)^r\frac{(2\ell-2r)!}{(\ell-r)!\,r!}\frac{x^{\ell-2r}}{(\ell-2r)!}$$

It is now clear that for any nonnegative integer $\ell$ even or odd, the polynomial

$$p_0(x) = k_\ell\sum_{r}(-)^r\frac{(2\ell-2r)!}{(\ell-r)!\,r!}\frac{x^{\ell-2r}}{(\ell-2r)!}$$

$$k_\ell \qquad (arbitrary)$$

is a solution of legendre’s equation.

# Rodrigues formula

$$p_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left(x^2 - 1\right)^n$$

The first few are

$$p_0(x) = 1$$

$$p_1(x) = x$$

$$p_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$p_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$p_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

## Example:

Evaluate $\left(1-t^2\right)\dfrac{d^2y}{dt^2}-2t\dfrac{dy}{dt}+6y=0$

---

## Solution:

Legendre's equation $\ell(\ell+1)=6$

i.e.

$$\ell^2+\ell-6=0$$

$$\therefore(\ell+3)(\ell-2)=0$$

$$\therefore\ell=2$$

The solution is

$$p_2(t) = \frac{1}{2}\left(2t^2 - 1\right)$$

But the general solution is

$$y(t) = a_0 y_1(t) + a_1 y_2(t)$$

Another form of legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dp}{d\theta}\right) + \ell(\ell+1)p = 0$$

## Example:

Evaluate $\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dz}{d\theta}\right]+2z=0$

Solution:

$$\ell(\ell+1)=2$$

$$\ell^2+\ell-2=0$$

$$(\ell+2)(\ell-1)=0$$

$$\therefore \ell=1$$

$$\therefore p_1(\cos\theta)=\cos\theta$$

The general solution is

$$z(\theta)=a_0\cos\theta+a_1 z_2(\theta)$$

From table

## Note:

That in the above form $x = \cos\theta$

$$\therefore\ p_2(\cos\theta) = \frac{1}{2}\left(3\cos^2\theta - 1\right)$$ and so on.

# Hermite-Functions

We shall outline some of the important formulas for two more sets of named functions. Both Hermit and laguerre functions are of interest in quantum mechanics where they arise as solutions of eigen value problems. We shall also consider an operator method which can be used as an alternative to series of some differential equations.

## Hermite function:

The differential equation for Hermite function is

$$y_n'' - x^2 y_n = -(2n+1)y_n \ , \quad n = 0,1,2,3,\ldots\ldots\ldots\ldots\ldots(1)$$

This equation can be solved by power series but here we shall consider an operator method which is particularly efficient for this equation.

Let us use the operator D to mean $\frac{d}{dx}$.

Then:

$$(D-x)(D+x)y=\left(\frac{d}{dx}-x\right)(y'+xy)=y''-x^2y+y \qquad \text{....................(2)}$$

And similarly

$$(D+x)(D-x)y=y''-x^2y-y \qquad \text{...............(3)}$$

Using (2) and (3) respectively we can write (1) in two ways:

$$(D-x)(D+x)y_n = -2ny_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(4) \; OR$$

$$(D+x)(D-x)y_n = -2(n+1)y_n \quad \ldots\ldots\ldots\ldots(5)$$

Now let us operate on (4) with $(D+x)$ and on (5) with $(D-x)$ and change $n$ to $m$ for later convenience .

$$(D+x)(D-x)[(D+x)y_m] = -2m[(D+x)y_m] \quad \ldots\ldots\ldots\ldots\ldots(6)$$

$$(D-x)(D+x)[(D-x)y_m] = -2(m+1)[(D-x)y_m] \quad \ldots\ldots\ldots(7)$$

Comparing (4) and (7) if $y_n = \left[(D-x)y_m\right]$ and

$n = m+1$ the equation are identical we write

$$y_{m+1} = (D-x)y_m \qquad \text{............}(8)$$

and we see that given a solution $y_m$ of (1) for one value of n, namely n = m we can find a solution for $n = m+1$ by applying the raising operator $(D-x)$ to $y_m$

Similarly from (5) and (6) we find that

$$y_{m-1} = (D+x)y_m \qquad \text{..............}(9)$$

We may call $(D+x)$ a" lowering operator "these operators are called creation and annihilation operators in quantum theory. Operators of this kind are called ladder operators.

---

i.e. / enable us to go up and down in a set of functions.

Now if n = 0 we find a solution of (4) and therefore of (1) by requiring $(D+x)y_0 = 0$.

We solve this equation to get $y_0 = e^{-x^2/2}$

$$\{(D+x)y_0 = 0$$

$$\frac{dy}{dx} + xy = 0$$

$$\ln y = \frac{-x^2}{2}$$

$$\therefore \int \frac{1}{y}\, dy = \int -x dx$$

$$\therefore y = e^{-x^2/2}\}$$

Then by (8)

$$y_n = (D - x)^n e^{-x^2/2}$$

These are the Hermite-Functions, they can be written in the simpler form:

$$y_n = e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \; . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

If we multiply (10) by $(-1)^n e^{x^2/2}$ we obtain the Hermite-polynomials may be called a Rodrigues formula for them

$$H_n(x) = (-1)^n e^{x^2} \frac{dn}{dx^n} e^{-x^2} \quad . \qquad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

We find

$$H_0(x) = 1 \qquad\qquad y_0 = e^{-x^2/2}$$

$$H_1(x) = 2x \qquad\qquad y_1 = (-1)2xe^{-x^2/2}$$

$$H_2(x) = 4x^2 - 2 \qquad\qquad y_2 = -(4x^2 - 2)e^{-x^2/2}$$

Where

$$H_n = (-1)^n e^{x^2/2} y_n$$

The Hermite polynomials satisfy the differential equation

$$y'' - 2xy' + 2ny = 0$$

Example:

Solve the differential equation $y'' - 2xy' + 20y = 0$

Solution:

$$n = 10 \qquad \therefore H_{10} = (-1)^{10} e^{x^2} \frac{d^{10}}{dx^{10}} e^{-x^2}$$

Using the differential equation, we can prove that the Hermite polynomials are orthogonal on $(-\infty, \infty)$ with respect to the weight function $e^{-x^2}$ the normalization integral can be evaluated thus we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) \quad H_m(x) \quad dx = \begin{cases} 0 & n \neq m \\ \sqrt{\pi}\, 2^n n! & n = m \end{cases}$$

---

The generating function for the Hermite polynomials is

$$\phi(x,h) = e^{-2xh-h^2} = \sum_{n=0}^{\infty} H_n(x) \frac{h^n}{n!}$$

The generating function can be used to drive recursion relations for the Hermite polynomials two useful relations are:

$$a) \quad H_n'(x) = 2nH_{n-1}(x)$$

$$b) \quad H_{n+1}(x) = 2nH_n(x) - 2nH_{n-1}$$

# Example:

Solve $y'' - x^2 y' = -5y$

Solution:

$$y'' - x^2 y' + 5y$$

$$\therefore 5 = 2n + 1$$

$$\therefore n = 2$$

The solution is

$$y_2 = (D - x)^2 y_0 = (D - x)^2 e^{-x^2/2}$$

# Laguerre Polynomials

The laguerre polynomials may be defined by a Rodrigues formula:

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \qquad \ldots\ldots\ldots\ldots(1)$$

Carrying out the differentiation using Leibniz Rule we find

$$L_n(x) = 1 - nx + \frac{n(n-1)}{2!}\frac{x^2}{2!} - \frac{n(n-1)(n-2)}{3!}\frac{x^3}{3!} + \ldots\ldots + \frac{(-1)^n x^n}{n!}$$

$$= \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{x^m}{m!} \quad Laguere \quad polynomial \qquad \ldots\ldots\ldots\ldots(2)$$

The symbol $\binom{n}{m}$ is a binomial coefficient

i.e. / $\binom{n}{m} = \dfrac{n!}{(n-m)!\,m!}$

From (2) we find that

$$L_0(x) = 1 \qquad *$$

$$L_1(x) = 1 - x \qquad ..$$

$$L_2(x) = 1 - 2x + \frac{x^2}{2}$$

The Laguerre polynomials are solutions of the differential equation: $xy'' + (1-x)y' + ny = 0$

i.e. / $y = L_n(x)$

called Laguerre Equation.

Using the differential equation we can prove that

the laguerre polynomials are orthogonal on $(0,\infty)$ with

respect to the weight function $e^{-x}$ .In fact we find that

with definition (1) the functions $e^{-x/2}L_n(x)$ are an

orthonormal set on $(0,\infty)$.

$$\int_0^\infty e^{-x}L_n(x)L_k(x)\quad dx=\begin{cases}0 & n\neq k\\ 1 & n=k\end{cases}\quad \text{.............}(3)$$

The generating function for the Laguerre polynomials is:

$$\phi(x,h)=\frac{e^{-xh/(1-h)}}{1-h}=\sum_{n=0}^{\infty}L_n(x)\quad h^n\quad \text{...........}(4)$$

Using it, we can derive recursion relations

Some examples are:

$$(a)\quad L'_{n+1}(x) - L'_n(x) + L_n(x) = 0$$

$$(b)\quad (n+1)L_{n+1}(x) - (2n+1-x)L_n(x) + nL_{n-1}(x) = 0$$

$$(c)\quad xL'_n(x) - nL_n(x) + nL_{n-1}(x) = 0$$

Derivatives of the Laguerre polynomials are called associated Laguerre polynomials they may be found by differentiating (1),(2) or *.

We define: $$L_n^k(x) = (-1)^k \frac{dk}{dx^k} L_{n+k}(x)$$

Associated Laguerre polynomials.

By differentiating the Laguerre equation we find the differential equation satisfied by the polynomials $L_n^k(x)$ is

$$xy'' + (k + -x)y' + ny = 0$$

$$y = L_n^k(x)$$

The polynomials $L_n^k(x)$ may be found from the Rodrigues formula

$$L_n^k(x) = \frac{x^{-k} e^x}{n!} \frac{d^m}{dx^n} \left( x^{n+k} e^{-x} \right)$$

Note in this form K does not have to be an integer.

Notice:

$$xy'' + (1 - x)y' + ny = 0$$

Using the series solution (Frobinous)

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$

and so on.

Example:

Solve the differential equation

$$xy'' + (1 - x)y' + 2y = 0$$

## Solution:

The solution is $L_2(x) = 1 - 2x + \frac{x^2}{2}$

---

## Example:

Using (2) Find $L_3(x)$ and $L_4(x)$

## Solution:

$$L_0(x) = 1$$

..

$$L_1(x) = 1 - x$$

$$L_2(x) = \tfrac{1}{2}\left(2 - 4x + x^2\right)$$

$$L_3(x) = \tfrac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)$$

$$L_4(x) = \tfrac{1}{24}\left(24 - 96x + 72x^2 - 16x^3 + x^4\right)$$

$$L_5(x) = \tfrac{1}{120}\left(120 - 600x + 600x^2 - 200x^3 + 25x^4 - x^5\right).$$

$$\sin x \quad \cos x = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$

$$\cos x \quad \cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$$

$$\sin x \quad \sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

# Fourier series or (orthogonal function):

$$F(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where $a_0$, $a_n$ and $b_n$ are constants.

$$\alpha \le x \le \alpha+2\pi \qquad (-\pi \le x \le \pi)$$

First we show the orthagonality property:

$$\int_{\alpha}^{\alpha+2\pi} \cos nx \quad \cos mx \quad dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}\{\cos(m+n)x+\cos(m-n)x\}dx.$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\right]_{\alpha}^{\alpha+2\pi} = 0$$

---

$n, m$ integer $m \neq n$

When $m = n$

$$\int_{\alpha}^{\alpha+2\pi} \cos nx \quad \cos mx \quad dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \{\cos(2n)x + 1\} dx$$

$$= \frac{1}{2}\left[\frac{\sin 2nx}{2n} + x\right]_{\alpha}^{\alpha+2\pi} = \frac{1}{2}[\alpha + 2\pi - \alpha] = \pi$$

Now replacing cos by sin

$$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \sin mx \quad dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi} \{\cos(m-n)x - \cos(m+n)x\} dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad m \neq n$$

When $m = n$ :

$$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \sin mx \quad dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}(1-\cos 2nx)dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = \pi$$

If we take a mixture of sin And cos :

$$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \cos mx \quad dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}\{\sin(m+n)x + \sin(m-n)x\}dx$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{(m-n)}\right]_{\alpha}^{\alpha+2\pi} = 0 \qquad m \neq n$$

When $m = n$:

$$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \cos mx \quad dx = \frac{1}{2} \int_{\alpha}^{\alpha+2\pi} \sin 2nx \quad dx.$$

$$= \frac{1}{2}\left[\frac{-\cos 2nx}{2n}\right]_{\alpha}^{\alpha+2\pi} = 0$$

## Summary:

(1) $$\int_{\alpha}^{\alpha+2\pi} \cos nx \quad \cos mx \quad dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

(2) $$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \sin mx \quad dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

(3) $$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad \cos mx \quad dx = \begin{cases} 0 & m \neq n \\ 0 & m = n \end{cases}$$

(4) $$\int_{\alpha}^{\alpha+2\pi} \cos nx \quad dx = \frac{1}{n}\left[\sin nx\right]_{\alpha}^{\alpha+2\pi} = 0$$

(5) $$\int_{\alpha}^{\alpha+2\pi} \sin nx \quad dx = \frac{1}{n}\left[-\cos nx\right]_{\alpha}^{\alpha+2\pi} = 0$$

(6) $$\int_{\alpha}^{\alpha+2\pi} dx = 2\pi$$

## Calculation of coefficients $a_0$, $a_n$ and $b_n$ :

$$\int_{\alpha}^{\alpha+2\pi} F(x)\,dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} dx + \sum_{n=1}^{\infty}\left\{a_n \int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx\,dx\right\}$$

$$= \frac{1}{2}a_0\, 2\pi + \sum_{n=1}^{\infty} a_n 0 + b_n 0$$

$$\therefore \pi\, a_0 = \int_{\alpha}^{\alpha+2\pi} F(x)\,dx. \quad *$$

$$\int_{\alpha}^{\alpha+2\pi} \cos mx\, F(x)\,dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty}\left\{a_n \int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\,dx + b_n \int_{\alpha}^{\alpha+2\pi} \cos mx \sin nx\,dx\right\}$$

$$\therefore \int_{\alpha}^{\alpha+2\pi} F(x) \quad \cos mx \quad dx = a_m \pi$$ *

$$\int_{\alpha}^{\alpha+2\pi} F(x) \quad \sin mx \quad dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} \sin mx \quad dx$$

$$+ \sum_{n=1}^{\infty} \left\{ a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \quad \sin mx \cdot dx + b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \quad \sin mx \quad dx \right\}$$

$$\therefore \int_{\alpha}^{\alpha+2\pi} F(x) \quad \sin mx \quad dx = b_m \pi$$ *

For the expanssion of $F(x)$: $equ(1)$ the parameters are given by:

$$\pi \quad a_0 = \int_{\alpha}^{\alpha+2\pi} F(x) \quad dx \quad *$$

$$\pi \quad a_n = \int_{\alpha}^{\alpha+2\pi} \cos nx \; . F(x) \quad dx \quad *$$

$$\pi \quad b_n = \int_{\alpha}^{\alpha+2\pi} \sin nx \quad F(x) \quad dx \quad *$$

This can be applied to discontinuous functions with the condition that $F(x_0) = \lim_{\varepsilon \to 0} \frac{1}{2}\{F(x_0+\varepsilon) + F(x_0-\varepsilon)\}$

Example:

$$F(x) = \cos px$$

$p$ is not an integer

$$-\pi \le x \le \pi$$

Solution:

$$\pi \quad a_0 = \int_{-\pi}^{\pi} \cos px \quad dx = \left[\frac{\sin px}{p}\right]_{-\pi}^{\pi} = \frac{2}{p}\sin p\pi$$

$(\sin p\pi \neq 0)$ since $p$ is not integer.

$$\pi \quad a_n = \int_{-\pi}^{\pi} \cos px \quad \cos nx \quad dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\cos(p+n)x + \cos(p-n)x\} dx$$

$$= \frac{1}{2}\left[\frac{\sin(p+n)x}{(p+n)} + \frac{\sin(p-n)x}{(p-n)}\right]_{-\pi}^{\pi} = \frac{\sin(p+n)\pi}{p+n} + \frac{\sin(p-n)\pi}{p-n}$$

$$\sin(p \pm n)\pi = \sin p\pi \quad \cos n\pi \pm \cos p\pi \quad \sin n\pi = (-1)^n \sin p\pi.$$

$$\therefore \pi \quad a_n = (-1)^n \sin p\pi \left\{\frac{1}{p+n} + \frac{1}{p-n}\right\}.$$

$$\pi \quad b_n = \int_{-\pi}^{\pi} \cos px \quad \sin nx \quad dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\sin(n+p)x + \sin(n-p)x\} dx$$

$$= -\frac{1}{2}\left[\frac{\cos(n+p)x}{(n+p)} + \frac{\cos(n-p)x}{(n-p)}\right]_{-\pi}^{\pi} = 0$$

For the expanssion

$$\cos px = \frac{\sin p\pi}{p\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} \sin p\pi \left( \frac{1}{p+n} + \frac{1}{p-n} \right)$$

$x = 0:$

$$\therefore \frac{\pi}{\sin p\pi} = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[ \frac{1}{p+n} + \frac{1}{p-n} \right]$$

*where:*

$$1 = \frac{1}{p\pi} \sin p\pi + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi} \sin p\pi \left( \frac{1}{p+n} + \frac{1}{p-n} \right)$$

# Example:

$$F(x) = x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \quad \cos nx + b_n \quad \sin nx\} \qquad \alpha \le x \le \alpha + 2\pi$$

# Solution:

$$\pi \quad a_0 = \int_{\alpha}^{\alpha+2\pi} F(x) \quad dx$$

$$\pi \quad a_n = \int_{\alpha}^{\alpha+2\pi} F(x) \quad \cos nx \quad dx$$

$$\pi \quad b_n = \int_{\alpha}^{\alpha+2\pi} F(x) \quad \sin nx \quad dx$$

$$\therefore \int x \quad \cos nx \quad dx = \frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx \qquad n \neq 0$$

$$= \frac{1}{2}x^2 \qquad n = 0$$

---

$$\int x \quad \sin nx \quad dx = \frac{-x}{n}\cos nx + \frac{1}{n^2}\sin nx \qquad n \neq 0$$

For the range $-\pi \leq x \leq \pi, \qquad \alpha = -\pi$ :

$$\pi \quad a_0 = \left[\frac{1}{2}x^2\right]_{-\pi}^{\pi} = 0$$

$$\pi \quad a_n = \left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0$$

$$\pi \quad b_n = \left[\frac{-x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi} = \frac{-2\pi\cos n\pi}{n}$$

$$\pi \quad b_n = \frac{2\pi}{n}(-1)^{n-1}$$

$$\therefore x = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx$$

$$(-\pi \le x \le \pi)$$

For the range $0 \le x \le 2\pi$:

$$\pi \quad a_0 = \left[\frac{1}{2}x^2\right]_0^{2\pi} = 2\pi^2$$

$$\pi \quad a_n = \left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{2\pi} = 0$$

$$\pi \quad b_n = \left[\frac{-x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi} = \frac{-2\pi}{n}\cos 2\pi n = \frac{-2\pi}{n}$$

$$\therefore x = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

Suppose $x = \pi/2$ then

$$\pi/2 = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi/2$$

$$\sin n\pi = \begin{array}{ll} 1 & n=1 \\ 0 & n=2 \\ -1 & n=3 \\ 0 & n=4 \end{array}$$

**Example:** $F(x) = 0 \qquad 0 \le x \le \pi$

$= \pi \qquad \pi \le x \le 2\pi$

**Solution:**

$$\pi \quad a_0 = \int_0^{2\pi} F(x) \quad dx = \pi \int_\pi^{2\pi} dx = \pi^2$$

$$\pi \quad a_n = \pi \int_\pi^{2\pi} \cos nx \quad dx = \frac{\pi}{n}\left[\sin 2n\pi - \sin n\pi\right] = 0$$

$$\pi \quad b_n = \pi \int_\pi^{2\pi} \sin nx \quad dx = \frac{-\pi}{n}\left[\cos 2n\pi - \cos n\pi\right] = \frac{-\pi}{n}\left[1 - (-1)^n\right]$$

$$\therefore F(x) = \frac{\pi}{2} - \frac{1}{n}\sum_{n=1}^{\infty}\left[1 - (-1)^n\right]\sin nx$$

value of $F(x)$ $at$ $x=\pi$

$$\lim_{\varepsilon \to 0} \frac{1}{2}\left\{F(\pi+\varepsilon)+F(\pi-\varepsilon)\right\} = \pi/2$$

$$\therefore \pi/2 - \sum_{n=1}^{\infty}\left(1-\frac{(-1)^n}{n}\right)\sin n\pi = \pi/2$$

# Condition for Fourier Expanssions for $F(x)$

1) $F(x)$ must be single valued except at a finite number of discontinuities.

2) $F(x)$ is periodic out side $(\alpha, \alpha + 2\pi)$ with period $2\pi$

3) $F(x)$ and $\dfrac{dF(x)}{dx}$ are section ally continuous in $(\alpha, \alpha + 2\pi)$

4) At discontinuous $(x_i)$ the series converges to

$$\lim_{\varepsilon \to 0} \frac{1}{2} \left\{ f(x_i + \varepsilon) + f(x_i - \varepsilon) \right\}.$$

These are called Dirichlet Conditions.

# Derivatives of Fourier series:

Example:

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx \qquad (-\pi \le x \le \pi)$$

Then

$$1 = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} n \quad \cos nx$$

$$2\sum_{n=1}^{\infty} (-1)^{n-1} \cos nx = 1$$

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n-1}$$

# Which doesn't make sense.

Given $$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \quad \cos nx + b_n \quad \sin nx\}.$$

$$\pi \quad a_0 = \int_{\alpha}^{\alpha+2\pi} f(x) \quad dx \qquad\qquad etc.$$

Then

$$\frac{df(x)}{dx} = f'(x) = \frac{a_0'}{2} + \sum_{n=1}^{\infty} \{a_n' \quad \cos nx + b_n' \quad \sin nx\}$$

$$\therefore \pi \quad a_0' = \int_{\alpha}^{\alpha+2\pi} f'(x) \quad dx = \left[f(x)\right]_{\alpha}^{\alpha+2\pi}$$

$$\pi \quad a_0' = f(\alpha + 2\pi) - f(\alpha)$$

$$\pi \quad a_n' = \int_{\alpha}^{\alpha+2\pi} f'(x) \quad \cos nx \quad dx$$

$$= \left[ f(x) \quad \cos nx \right]_{\alpha}^{\alpha+2\pi} + n \int_{\alpha}^{\alpha+2\pi} f(x) \quad \sin nx \quad dx$$

$$= f(\alpha + 2\pi)\cos n(\alpha + 2\pi) - f(\alpha) \quad \cos n\alpha + n \int_{\alpha}^{\alpha+2\pi} f(x) \quad \sin nx \quad dx$$

$$\pi \quad a_n' = \cos n\alpha \left[ f(\alpha + 2\pi) - f(\alpha) \right] + n b_n \pi$$

$$\pi \quad b_n' = \int_{\alpha}^{\alpha+2\pi} f'(x) \quad \sin nx \quad dx$$

$$=\left[f(\alpha+2\pi)-f(\alpha)\right]\sin n\alpha-n\int_{\alpha}^{\alpha+2\pi}f(x)\quad\cos nx\quad dx$$

$$\pi\quad a_n'=\cos n\alpha\left[f(\alpha+2\pi)-f(\alpha)\right]+nb_n\pi$$

$$\pi\quad b_n'=\int_{\alpha}^{\alpha+2\pi}f'(x)\quad\sin nx\quad dx$$

$$=\left[f(\alpha+2\pi)-f(\alpha)\right]\sin n\alpha-n\int_{\alpha}^{\alpha+2\pi}f(x)\quad\cos nx\quad dx$$

$$\pi\quad b_n'=\left[f(\alpha+2\pi)-f(\alpha)\right]\sin n\alpha-na_n\pi$$

We check that work by applying it to:

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx \qquad (-\pi \le x \le \pi)$$

$$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{2}{n}(-1)^{n-1}$$

$$\therefore \pi \quad a_0' = [\alpha + 2\pi - \pi] = 2\pi$$

$$\pi \quad a_n' = \cos n\pi[2\pi] + n\pi \frac{2}{n}(-1)^{n-1} = 0$$

$$\pi \quad b_n' = \sin n\pi[2\pi] - 0 = 0$$

# Fourier Series Expanssions of even or odd

## Functions in $-\pi \le x \le \pi$

Even function $f(-x) = f(x)$

Odd function $f(-x) = -f(x)$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

$$\pi a_0 = \int_{-\pi}^{\pi} f(x)\,dx = \int_{\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \int_{\pi}^{0} f(-y)(-dy) + \int_{0}^{\pi} f(x)\,dx$$

$$\therefore \pi a_0 = \int_{0}^{\pi} [f(x) + f(-x)]dx$$

$$\pi \quad a_n = \int_{-\pi}^{\pi} \cos nx \quad f(x) \quad dx$$

$$\int_{-\pi}^{0} \cos nx \quad f(x) \quad dx = \int_{\pi}^{0} \cos ny \quad f(-y) \quad (-dy) = \int_{0}^{\pi} \cos nx \quad f(-x) \quad dx$$

$$\therefore \pi \quad a_n = \int_{0}^{\pi} \cos nx [f(x) + f(-x)] dx$$

$$\pi \quad b_n = \int_{-\pi}^{\pi} \sin nx \quad f(x) \quad dx$$

$$\int_{-\pi}^{0} \sin nx \quad f(x) \quad dx = \int_{\pi}^{0} \sin(-ny) \quad f(-y) \quad -dy = -\int_{0}^{\pi} \sin nx \quad f(-x) \quad dx$$

$$\therefore \pi \quad b_n = \int_0^{\pi} \sin nx \left[ f(x) - f(-x) \right] dx$$

i) If $f(x)$ even function i.e. / $f(-x) = f(x)$

Then

$$\pi \quad a_0 = 2\int_0^{\pi} f(x) \quad dx$$

$$\pi \quad a_n = 2\int_0^{\pi} \cos nx \quad f(x) \quad dx$$

$$\pi \quad b_n = 0$$

i.e. / $$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

ii) If $f(x)$ odd function i.e. / $f(-x) = -f(x)$

Then $a_0 = 0$ $a_n = 0$

$$\pi \quad b_n = 2\int_0^{\pi} \sin nx \quad f(x) \quad dx$$

i.e. /

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

# Example:

Find the Fourier series

$$f(x) = x^2 \qquad -\pi < x < \pi$$

$$f(x) = (\alpha + 2\pi)$$

# Solution:

The function is even

$$\therefore b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \quad dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \quad \cos nx \quad dx$$

Let $u = x^2 \Rightarrow du = 2x\,dx$

i.e. / $u = x^2$ $\quad \dfrac{du}{dx} = 2x$ $\quad \dfrac{dv}{dx} = \cos nx$ $\quad v = \dfrac{\sin nx}{n}$

$$\therefore a_n = \frac{2}{\pi}\left[\left(\frac{x^2 \sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} \frac{2x \quad \sin nx}{n} \quad dx\right]$$

..= 0

$u = x$ $\quad \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \sin nx$ $\quad v = \dfrac{-\cos nx}{n}$

$$= \frac{2}{\pi}\left[\frac{-2}{n}\left(\frac{-x \quad \cos nx}{n}\right)_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} \quad dx\right]$$

..= 0

$$= \frac{2}{\pi} . \frac{2\pi}{n^2} \cos n\pi = \frac{4}{n^2} (-1)^n$$

---

$$\therefore \; f(x) = x^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

i.e. /

$$x^2 = \frac{2\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \ldots\ldots\ldots$$

$$= \frac{2\pi^2}{3} - 4\cos x + \cos 2x - \frac{4}{9}\cos 3x + \ldots\ldots\ldots\ldots\ldots$$

$$= \frac{2\pi^2}{3} - 4\left[\cos x - \cos 2x + \frac{\cos 3x}{9} + \ldots\ldots\right]$$

$$x^2 - \frac{2\pi^2}{3} = -4\left[\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - .........\right]$$

If $x = \pm\pi$ Then:

$$\frac{\pi^2}{3} = 4\left[1 + \frac{1}{4} + \frac{1}{9} + ......\right]$$

$$\frac{\pi^2}{12} = 1 + \frac{1}{4} + \frac{1}{9} + ....... = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If $x = 0$ Then:

$$0 - \frac{2\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - ....\right] = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

## Example:

$$f(x) = \begin{cases} -\cos x & -\pi < x < \pi \\ \cos x & 0 < x \le \pi \end{cases}$$

$$f(\alpha + 2\pi) = f(\alpha)$$

## Solution:

The function is odd even though *cos* is even.

$$\therefore a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \quad \sin nx \quad dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \quad \sin nx \quad dx$$

$$= \frac{2}{\pi} . \frac{1}{2} \int_0^{\pi} \left[ \sin(n+1)x + \sin(n-1)x \right] dx$$

$$= \frac{-1}{\pi} \left[ \frac{\cos(n+1)\pi}{(n+1)} - \frac{1}{n+1} + \frac{\cos(n-1)\pi}{(n-1)} - \frac{1}{n-1} \right]$$

$$\cos(n+1)\pi = (-1)^{n+1} = -(-1)^n$$

$$\cos(n-1)\pi = (-1)^{n-1} = -(-1)^n$$

$$\therefore b_n = \frac{-1}{\pi} \left[ \frac{-(-1)^n}{n+1} + \frac{-(-1)^n}{n-1} - \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{1}{\pi} \left[ (-1)^n \left( \frac{1}{n+1} + \frac{1}{n-1} \right) + \frac{1}{n+1} + \frac{1}{n-1} \right]$$

$$\therefore b_n = \frac{1}{\pi}\left[\left\{(-1)^n + 1\right\}\left(\frac{2n}{n^2 - 1}\right)\right]$$

---

$$\therefore f(x) = \frac{4}{\pi}\sum_{n=2,4,6,\ldots}\frac{n}{n^2 - 1}\sin nx$$

$where \quad n \quad even \qquad (-1)^n = 1 \qquad (-1)^n + 1 = 2$

$\qquad n \quad odd \qquad (-1)^n = -1 \qquad (-1)^n + 1 = 0$

# Fourier's Expanssions in Range $0 \le x \le \pi$

These are of the alternative types

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

OR:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Out side $(0, \pi)$

Cosine series must be even in range $-\pi \le x \le \pi$

Sine series must be odd in range $-\pi \le x \le \pi$

# Example:

$$f(x) = x \text{ expanded as cosine series}$$

i.e. /

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \quad \cos nx$$

$$\pi \quad a_1 = 2\int_0^{\pi} x \quad dx = \pi^2 \quad ..$$

$$\pi \quad a_n = 2\int_1^{\pi} x \quad \cos nx \quad dx = 2\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{n^2}\left[(-1)^n - 1\right]$$

$$\therefore x = \frac{\pi}{2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2}\left[(-1)^n - 1\right]\cos nx$$

# Example:

$f(x) = x$ expanded as sine series in range $0 \le x \le \pi$

$$\pi \quad b_n = 2\int_0^{\pi} x \quad \sin nx \quad dx$$

$$= 2\left[\frac{-x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$

$$= \frac{-2\pi}{n}\left[(-1)^n - 1\right]$$

$$\therefore x = 2\sum_{n=1}^{\infty} \frac{1}{n}\left[1 - (-1)^n\right]\sin nx$$

# Fourier's Expanssions in Range $-\ell \le x \le \ell$

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left\{A_n \cos\frac{n\pi x}{\ell} + B_n \sin\frac{n\pi x}{\ell}\right\}$$

$$\int_{-\ell}^{\ell} dx = 2\ell$$

$$\int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\,dx = 0 = \int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\,dx$$

$$\int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\cos\frac{m\pi x}{\ell}\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{\cos(m+n)\frac{\pi x}{\ell} + \cos(m-n)\frac{\pi x}{\ell}\right\}dx.$$

$$= 0 \qquad (m \neq n)$$

$$= \frac{1}{2}\int_{-\ell}^{\ell}\left(\cos\frac{2nx}{\ell} + 1\right)dx = \frac{1}{2}(2\ell)$$

$$= \ell \qquad (m = n)$$

$$\int_{-\ell}^{\ell} \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \frac{1}{2}\int_{-\ell}^{\ell}\left\{\cos(m-n)\frac{\pi x}{\ell} - \cos(m+n)\frac{\pi x}{\ell}\right\}dx$$

$$= 0 \quad (m \neq n)$$

$$= \ell \quad (m = n)$$

$$\int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell}\cos\frac{n\pi x}{\ell}\,dx = 0$$

---

$$\ell \quad A_0 = \int_{-\ell}^{\ell} F(x) \quad dx$$

$$\ell \quad A_n = \int_{-\ell}^{\ell} F(x) \quad \cos\frac{n\pi x}{\ell} \quad dx$$

$$\ell \quad B_n = \int_{-\ell}^{\ell} F(x) \quad \sin\frac{n\pi x}{\ell} \quad dx$$

# Fourier's Expansion in Range $(0,\ell)$

Either:

$$F(x)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{\ell}$$

$$\ell\quad A_0=2\int_0^{\ell}F(x)\,dx$$

$$\ell\quad A_n=2\int_0^{\ell}F(x)\cos\frac{n\pi x}{\ell}\,dx$$

$\left(\begin{matrix}Even\\Function\end{matrix}\right)$

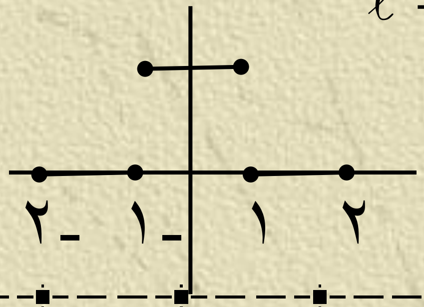
OR:

$$\left.\begin{aligned} F(x) &= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \\ \ell \quad B_n &= 2\int_0^{\ell} F(x) \sin\frac{n\pi x}{\ell}\, dx \end{aligned}\right\} \begin{pmatrix} Odd \\ Function \end{pmatrix}$$

# Example:

$\ell = 4$

Periodic square wave




$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

# Solution:

..

The function is even symmetry about they-axis.

i.e. / $B_n = 0$

$$\therefore \ell \quad A_0 = \int_{-2}^{2} F(x) \quad dx = \int_{-2}^{2} k dx = 2\int_{0}^{2} k dx$$

$$= 2\int_0^1 k\,dx + \int_1^2 0\,dx = 2[kx]_0^1 = 2k$$

$$\therefore A_0 = \frac{2k}{4} = \frac{1}{2}k$$

$$\ell \quad A_n = \int_{-\ell}^{\ell} F(x) \quad \cos\frac{n\pi x}{\ell} \quad dx$$

$$\therefore A_n = \frac{1}{4}\int_{-2}^{2} k\cos\frac{n\pi x}{2} \quad dx$$

$$= \int_0^1 k\cos\frac{n\pi x}{2} \quad dx + \int_1^2 0\cos\frac{n\pi x}{2} \quad dx$$

$$= \left[ \frac{2k}{n\pi} \sin \frac{n\pi x}{2} \right]_0^1 = \frac{2k}{n\pi} \sin \frac{n\pi}{2}$$

---

i.e. /

$$A_n = \frac{2k}{n\pi} (-1)^{\frac{n-1}{2}} = 0 \qquad (even \quad n)$$

$$= \frac{2k}{n\pi} \qquad n = 1, 5, 9, \ldots\ldots\ldots$$

$$= \frac{-2k}{n\pi} \qquad n = 3, 7, 11, \ldots\ldots\ldots$$

$$\therefore\ f(x) = \frac{1}{4}k + \frac{2k}{\pi} \cos \frac{\pi x}{2} - \frac{2k}{3\pi} \cos \frac{3\pi}{4} x + \ldots\ldots\ldots$$

# Partial Differential Equations

## Introduction:

Any equation containing partial differential coefficients is called a partial differential equation. The order of the equation being equal to the order of the highest partial differential coefficient occurring in it.

e.g. /

$$3y^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2u$$

$$\frac{\partial^2 u}{\partial x^2} + f(x, y)\frac{\partial^2 u}{\partial y^2} = 0$$

Where $f(x, y)$ is an arbitrary function are typical partial differential equations of the 1st and 2nd orders respectively

$x$ and $y$ being independent variables and $u$ the function to be found, and both are linear.

A typical non-linear equation in two independent variables is

$$u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial y}\right)^2=u^2.$$

The general solution of an ordinary differential equation contains arbitrary constants. The general solution of a partial differential equation, however usually contains arbitrary functions.

Let us consider $u = y \quad f(x).$ ............(1)

Where $f(x)$ is an arbitrary function of $x$ then

$$\frac{\partial u}{\partial y} = f(x). \qquad \text{................}(2)$$

Eliminating $f(x)$ we obtain 1st order P.D.E.

$$y\frac{\partial u}{\partial y} = u. \qquad \text{...............}(3)$$

$\therefore$ Any function $u$ of the type given by (1) is a solution of (3)

---

Similarly if $$u = f(x+y) + g(x-y) \quad *$$

Where $f(x+y)$, $g(x-y)$ are arbitrary functions of $x+y$ and $x-y$ respectively then

$$\frac{\partial u}{\partial x} = f'(x+y) + g'(x-y) \qquad \text{............}(4)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+y) + g''(x-y) \qquad \text{............}(5)$$

$$\frac{\partial u}{\partial y} = f'(x+y) - g'(x-y) \qquad \ldots\ldots\ldots\ldots(6)$$

---

$$\frac{\partial^2 u}{\partial y^2} = f''(x+y) + g''(x-y) \qquad \ldots\ldots\ldots\ldots(7)$$

Hence from equ (5) and (7)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad \ldots\ldots\ldots\ldots(8)$$

The function $u$ defined in * therefore satisfies (8).

# Second Order Constant Coefficient Equations

Any equation of the type

$$a\frac{\partial^2 u}{\partial x^2} + 2h\frac{\partial^2 u}{\partial x \partial y} + b\frac{\partial^2 u}{\partial y^2} + 2f\frac{\partial u}{\partial x} + 2g\frac{\partial u}{\partial y} + cu = 0 \qquad \ldots\ldots\ldots.(9)$$

Where *a,h,b,f,g* and *c* are constants, is a linear 2nd order constant coefficient P.D.E. ∵ in two variables( $x$ and $y$) by comparison with the equation of the general conic

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + c = 0 \qquad \ldots\ldots\ldots\ldots(10)$$

We say that (9) is of

$$\left.\begin{array}{l} Elliptic \\ parabolic \\ hyperbolic \end{array}\right\} type \quad when \begin{cases} ab - h^2 > 0 \\ ab - h^2 = 0 \\ ab - h^2 < 0 \end{cases}$$

---

e.g. / Lap lace's equation in two variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

May be obtained from (9) by putting

$$a = 1, \quad b = 1, \quad h = f = g = c = 0$$

$\therefore ab - h^2 > 0$ is of elliptic type.

Similarly,

$$\frac{\partial^2 u}{\partial x^2} - k^2 \frac{\partial^2 u}{\partial y^2} = 0$$

(Where $k$ is a real constant) may obtained from (9) by

putting $a = 1, \quad b = -k^2, \quad h = f = g = c = 0$

--------------------------------------------

$\therefore ab - h^2 = -k^2 < 0$

the equation is of hyperbolic type.

..

However the equation

$$\frac{\partial^2 u}{\partial x^2} - k\frac{\partial u}{\partial y} = 0$$

is of parabolic type since $a = 1, \quad h = b = 0$

$$g = -\tfrac{1}{2}k, \quad f = c = 0, \quad and \quad ab - h^2 = 0$$

---

## Euler's Equation

The equation $a\dfrac{\partial^2 u}{\partial x^2} + 2h\dfrac{\partial^2 u}{\partial x \partial y} + b\dfrac{\partial^2 u}{\partial y^2} = 0$ .........(11)

Where *a,h* and *b* are constants. Is a special case of (9)

$(f = g = c = 0)$ known as Euler's equation.

The general solution of this equation obtained as follows:

Let $\varepsilon = px + qy$

$$\eta = rx + sy$$

Where $p, q, r$ *and* $s$ are arbitrary constants, then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = p\frac{\partial u}{\partial \varepsilon} + r\frac{\partial u}{\partial \eta}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \varepsilon}.\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y} = q\frac{\partial u}{\partial \varepsilon} + s\frac{\partial u}{\partial \eta}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(p\frac{\partial}{\partial \varepsilon} + r\frac{\partial}{\partial \eta}\right)\left(p\frac{\partial u}{\partial \varepsilon} + r\frac{\partial u}{\partial \eta}\right)$$

$$= p^2\frac{\partial^2 u}{\partial \varepsilon^2} + 2pr\frac{\partial^2 u}{\partial \varepsilon \partial \eta} + r^2\frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(q\frac{\partial}{\partial \varepsilon} + s\frac{\partial}{\partial \eta}\right)\left(q\frac{\partial u}{\partial \varepsilon} + s\frac{\partial u}{\partial \eta}\right)$$

$$= q^2\frac{\partial^2 u}{\partial \varepsilon^2} + 2sq\frac{\partial^2 u}{\partial \varepsilon \partial \eta} + s^2\frac{\partial^2 u}{\partial \eta^2}.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(p\frac{\partial}{\partial \varepsilon} + r\frac{\partial}{\partial \eta}\right)\left(q\frac{\partial u}{\partial \varepsilon} + s\frac{\partial u}{\partial \eta}\right)$$

$$= pq\frac{\partial^2 u}{\partial \varepsilon^2} + (rq + sp)\frac{\partial^2 u}{\partial \varepsilon \partial \eta} + rs\frac{\partial^2 u}{\partial \eta^2}.$$

$\because$ Substituting in equ(11) we get

$$\left(ap^2+bq^2+2hpq\right)\frac{\partial^2 u}{\partial \varepsilon^2}+2\left\{apr+bsq+h\left(rq+sp\right)\right\}\frac{\partial^2 u}{\partial \varepsilon \partial \eta}$$

$$+\left(ar^2+bs^2+2hrs\right)\frac{\partial^2 u}{\partial \eta^2}=0. \qquad \text{..........}(12)$$

We now choose $p=r=1$ and $q$ and s are the two roots $X_1$ *and* $X_2$ of the equation

$$a+2hX+bX^2=0$$

$\therefore (12)$ *becomes*

$$\left\{a+h\left(X_1+X_2\right)+bX_1X_2\right\}\frac{\partial^2 u}{\partial \varepsilon \partial \eta}=0$$

$$\therefore X_1 + X_2 = \frac{-2h}{b}$$

---

$$X_1 X_2 = \frac{a}{b}$$

i.e. / $\frac{2}{b}\left(ab - h^2\right) = \frac{\partial^2 u}{\partial \varepsilon \partial \eta} = 0$ ............(13)

Provided (11) is not parabolic in the sense that

$ab - h^2 \neq 0$ then (13) gives $\frac{\partial^2 u}{\partial \varepsilon \partial \eta} = 0$

By integration the general solution

$$u = F(\varepsilon) + G(\eta)$$

Where $F$ and $G$ are arbitrary functions.

Since $\varepsilon = x + X_1 y, \quad \eta = x + X_2 y$ the general solution of (11) is

$$u = F(x + X_1 y) + G(x + X_2 y)$$

.(Provided (11) is not of parabolic type)

Finally the nature of the roots $X_1$ *and* $X_2$ depends on

whether the equ(11) is of hyperbolic or elliptic type for .

when $ab - h^2 < 0$(hyperbolic) $X_1$ *and* $X_2$ are real.

When $ab - h^2 > 0$ (Elliptic) $X_1$ *and* $X_2$ are complex.

When (11) is of parabolic type $\left(ab - h^2 = 0\right)$ then the general solution obtained from (12) by putting $p = 1$

Then:

$$\left(a+bq^2+2hq\right)\frac{\partial^2 u}{\partial \varepsilon^2}+2\left\{ar+bsq+h\left(rq+s\right)\right\}\frac{\partial^2 u}{\partial \varepsilon \partial \eta}$$

$$+\left(ar^2+bs^2+2hrs\right)\frac{\partial^2 u}{\partial \eta^2}=0 \qquad \ldots\ldots\ldots\ldots(14)$$

If $q$ is now chosen to be the root of

$$a+bq^2+2hq=0$$

Since $ab-h^2=0$ then $q=\frac{-h}{b}$ (twice).

Hence

$$ar+bsq+h\left(rq+s\right)=\left(ab-h^2\right)\frac{r}{b}=0$$

$\therefore (14)$ becomes

$$\frac{\partial^2 u}{\partial \eta^2} = 0 \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

By direct integration the general solution of (15)

$$u = F(\varepsilon) + \eta G(\varepsilon)$$

Since

$$p = 1, \quad q = \frac{-h}{b} \quad \because \; (= X \quad say)$$

We have

$$\varepsilon = x + Xy$$

$$\eta = rx + sy$$

The solution is therefore

$$u = F(x + Xy) + (rx + sy)G(x + Xy)$$

Which is the general solution of (11) when

$$ab - h^2 = 0$$

وصلى الله على سيدنا محمد وعلى آله وصحبه وسلم.